SHORTER NOTES

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A SUFFICIENT CONDITION THAT THE LIMIT OF A SEQUENCE OF CONTINUOUS FUNCTIONS BE AN EMBEDDING

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ABSTRACT. Suppose X is a metric space, and Y is a complete metric space. In this paper a sufficient condition is given to insure that a sequence of continuous functions from X into Y converge to an embedding from X into Y.

In this note we give a sufficient condition that the limit of a sequence of continuous functions be an embedding. This result is a sharpening of one of Bing's results [1].

The difference between Bing's theorem and the one given here is that we remove the requirement that X be compact and relax the requirement of bicontinuity on the f_n's to continuity, in particular the f_n's need not even be 1-1. In the following theorem, by $\rho(f, g)$ we shall mean $\sup_{t\in X} \rho(f(t), g(t))$.

THEOREM. *Suppose $\{f_n\}$ is a sequence of continuous functions from a metric space X into a complete metric space Y and suppose that $\{\epsilon_i\}$ is a sequence of numbers such that $\rho(f_i, f_{i+1}) < \epsilon_{i+1}$ for each i, and such that $\rho(x_1, x_2) > 1/j$ implies $\rho(f_j(x_1), f_j(x_2)) > 2\sum_{i=j}^{\infty} \epsilon_i$ for all $x_1, x_2 \in X$. Then $f = \lim_n f_n$ is an embedding.*

PROOF OF THEOREM. That f is continuous follows because it is the uniform limit of continuous functions.

Suppose that $y_n \to y_0$ in $f[X] \subset Y$. For each n, $n = 0, 1, 2, \cdots$, we chose an $x_n \in X$ such that $f(x_n) = y_n$. Assume that x_n fails to converge to x_0. Then, it follows that there is an $\epsilon_0 > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\rho(x_{n_i}, x_0) > \epsilon_0$ for $i = 1, 2, \cdots$. We choose N such that $1/N < \epsilon_0$. Therefore, it follows from our hypothesis that $\rho(f_N(x_{n_i}), f_N(x)) > 2\sum_{i=N}^{\infty} \epsilon_i$ for each i since $\rho(x_{n_i}, x_0) \geqq \epsilon_0 > 1/N$ for each i. Since $f(x_{n_i}) = y_{n_i} \to y_0 = f(x_0)$, then there is a k such that $\rho(f(x_{n_k}), f(x_0)) < \epsilon_N$.

We observe that

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$$\rho(f_N, f) = \lim_n \rho(f_N, f_{N+n}) \leqq \lim_n \sum_{i=N+1}^{N+n} \epsilon_i = \sum_{i=N+1}^{\infty} \epsilon_i.$$

Therefore,

$$\rho(f_N(x_{n_k}), f_N(x_0)) \leqq \rho(f_N(x_{n_k}), f(x_{n_k})) + \rho(f(x_{n_k}), f(x_0)) + \rho(f(x_0), f_N(x_0)) < 2 \sum_{i=N+1}^{\infty} \epsilon_i + \epsilon_N.$$

But this contradicts the condition $\rho(f_N(x_{n_k}), f_N(x_0)) > 2\sum_{i=N}^{\infty} \epsilon_i$, which is guaranteed by our choice of $\{x_{n_i}\}$. Therefore, we are led to conclude that $x_n \to x_0$.

The above paragraph implies the inverse of f is continuous if the inverse is a single valued function from $f[X]$ onto X, i.e., if f is 1-1. However, that f is 1-1 can also be deduced from the previous paragraph, for if $f(x) = f(x') = y_0$, choose $y_n = y_0$ for $n = 1, 2, \cdots$ and choose the sequence $x_n = x$ for $n = 1, 2, \cdots$ and $x_0 = x'$; it follows from the previous paragraph that $x_n \to x_0$ which implies $x = x'$.

Therefore, f is 1-1 and bicontinuous; hence, an embedding.

References

1. R. H. Bing, *Each disk in E^3 contains a tame arc*, Amer. J. Math. **84** (1962), 583–590. MR **26** #4331.

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