STRICTLY CONVEX SPACES VIA SEMI-INNER-PRODUCT SPACE ORTHOGONALITY[1]

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ABSTRACT. Let $(X, \|\cdot\|)$ be a normed space, and let $[\cdot, \cdot]$ be any semi-inner-product on it. We show that $(X, \|\cdot\|)$ is strictly convex if and only if $\|y+z\| > \|y\|$ whenever $[z, y]=0$ and $z \neq 0$, and if and only if $[Ax, x] \neq 0$ whenever $\|I+A\| \leqq 1$ and $Ax \neq 0$. The condition that $[z, y]=0$ can be replaced by a stronger or weaker condition.

A complex or real normed linear space $(X, \|\cdot\|)$ is strictly convex if each point of the unit sphere is an extreme point of the unit ball. Every normed space has at least one semi-inner-product [**4**, Theorem 2, p. 31], i.e., a map $[\cdot, \cdot]$ on $X \times X$ to C (resp., to R) such that

(i) $[\lambda x+y, z]=\lambda[x, z]+[y, z]$,

(ii) $[x, x]=\|x\|^2$,

(iii) $|[x, y]| \leqq \|x\|\|y\|$,

for all x, y, z in X, λ in C (resp., λ in R).

For a given semi-inner-product $[\cdot, \cdot]$ on the space, one can say that y is orthogonal to z if $[z, y]=0$; the condition that y is orthogonal to z then depends on the choice of semi-inner-product. Nonetheless, we show that if $[\cdot, \cdot]$ is any semi-inner-product on $(X, \|\cdot\|)$, the space is strictly convex if and only if $\|y+z\| > \|y\|$ whenever y is orthogonal to $z \neq 0$, and if and only if x is never orthogonal to $Ax \neq 0$ for operators A such that $\|I+A\| \leqq 1$.

This result still holds if the original orthogonality is replaced by a stronger or weaker form, both of which depend only on the normed space, and the latter of which is equivalent to that of James [**3**, p. 265].

The last paragraph contains an application of condition (iv).

Related results have been obtained by Berkson [**1**, Theorem 5.1, p. 381, and Lemma 5.3, p. 382], James [**3**, Theorem 4.3, p. 275, and Theorem 5.2, p. 279], and Palmer [**5**, p. 4].

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THEOREM. *Let* $[\cdot, \cdot]$ *be any semi-inner-product on* $(X, \|\cdot\|)$. *The following conditions are equivalent*:

(i) $(X, \|\cdot\|)$ *is strictly convex*;

(ii) *If* $\|y+z\| \leqq \|y\|$ *and* $[z, y]=0$, *then* $z=0$;

(iii) *If* $\|y+z\| = \|y\|$ *and* $[z, y]=0$, *then* $z=0$;

(iv) *If* A *is a bounded linear operator on* X, *if* $\|I+A\| \leqq 1$, *and if* $[Ax, x]=0$ *for some* x *in* X, *then* $Ax=0$.

PROOF. It will be shown that (i)$\Rightarrow$(ii)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(i).

(i)$\Rightarrow$(ii). Let $(X, \|\cdot\|)$ be strictly convex, $\|y+z\| \leqq \|y\|$, and $[z, y] =0$. We may assume that $y\neq 0$. For $0\leqq t\leqq 1$ we have

$$\begin{aligned}\|y\|^2 = |[y, y]| &= |[y+tz, y]| \leqq \|y+tz\|\,\|y\| \\ &\leqq (t\|y+z\| + (1-t)\|y\|)\|y\| \leqq \|y\|^2,\end{aligned}$$

whence $\|y+tz\| = \|y\|$ for all t, $0\leqq t\leqq 1$. Since $(X, \|\cdot\|)$ is strictly convex, it follows that $z=0$.

(ii)$\Rightarrow$(iv). Trivial.

(iv)$\Rightarrow$(iii). Suppose that (iii) does not hold; then there exist $y\neq 0$, $z\neq 0$ such that $\|y+z\| = \|y\|$ and $[z, y]=0$. Let the bounded operator A be defined as follows:

$$Ax = (1/\|y\|^2)[x, y](y+z) - x, \qquad x\in X.$$

Then $\|I+A\| \leqq 1$, $Ay=z\neq 0$ and $[Ay, y]=[z, y]=0$; thus condition (iv) does not hold.

(iii)$\Rightarrow$(i). Suppose that the space is not strictly convex. Then there exist distinct points u, v and w in X such that $\|u\|=1$, $\|v\|\leqq 1$, $\|w\|\leqq 1$, and u is on the line segment between v and w. Clearly, $\|v\| = \|w\| = 1$. Let $y=(v+w)/2$ and $z=v-y=(v-w)/2$. Since $\|y\|\leqq 1$, $\|u\|=1$, and u is on either the segment between v and y or the segment between y and w, it follows that $\|y\|=1$. Thus

$$|[z, y]+1| = |[v-y, y]+[y, y]| = |[v, y]| \leqq 1,$$

and

$$|[z, y]-1| = |[z-y, y]| = |[-w, y]| \leqq 1,$$

from which it follows that $[z, y]=0$. Since $z\neq 0$ and $\|y+z\| = \|v\| = 1 = \|y\|$, (iii) does not hold.

COROLLARY. *The condition that* $[z, y]=0$ *can be replaced by the stronger condition that* $([z, y]) = \{0\}$, *where*

$$\begin{aligned}([z, y]) &= \{y^*z : y^*\in X^* \ \text{ and } \ y^*y = \|y\|^2 = \|y^*\|^2\} \\ &= \{[z, y] : [\cdot, \cdot] \text{ is a semi-inner-product on } (X, \|\cdot\|)\}.\end{aligned}$$

Similarly, "$[z, y]=0$" *can be replaced by* "$0\in([z, y])$," *and* "$[Ax, x]=0$" *by* "$0\in([Ax, x])$."

PROOF. Some implications are trivial; the others can be seen by a careful reading of the previous proof.

REMARK. The condition that $0\in([z, y])$ is equivalent to a definition of orthogonality due to James [**3**, p. 265 and Theorem 2.1, p. 268], who noted that in spaces of dimension greater than or equal to three, this orthogonality is symmetric if and only if the norm is given by an inner product [**2**, Theorem 1, p. 560].

Application. Some operators to which condition (iv) is relevant are those which are normal elements of some B^*-algebra, and whose spectrum lies inside some disc which does not contain 0 as an interior point. For if T is such an operator and D is the unit disc, then $\sigma(T)$ is contained in some disc $\lambda D-\lambda$, and $\sigma(I+(1/\lambda)T)\subseteq D$, whence by [**6**, Lemma 4.8.1 i, p. 240] we have

$$\|I + (1/\lambda)T\| = \|I + (1/\lambda)T\|_\sigma \leqq 1.$$

Thus if $[Tx, x]=0$ then $[(1/\lambda)Tx, x]=0$, whence $Tx=0$.

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