DIVISIBLE H-SPACES[1]

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ABSTRACT. Let X be an H-space with multiplication m. Define, for $x\in X$, $m_2(x)=m(x, x)$ and $m_k(x)=m(x, m_{k-1}(x))$, for all $k>2$. If $m_k(x)=y$, then x is called a kth root of y. The H-space (X, m) is divisible if every y in X has a kth root for each $k\geqq 2$. We prove that if X is a compact connected topological manifold without boundary, then (X, m) is divisible and, in fact, that every y in X has at least k^β kth roots for each $k\geqq 2$, where β is the first Betti number of X.

A group G is *divisible* if given any $g\in G$ and any integer $k\geqq 2$, there is a solution to $x^k=g$ in G. A solution x is called a kth *root* of g. In 1940, Hopf proved that a compact connected Lie group G is divisible and, moreover, that for each integer $k\geqq 2$, every $g\in G$ has either k^λ kth roots or an infinite number, where λ is the number of generators of the exterior algebra $H^*(G)$ (rational coefficients) [**4**].

By an *H-space*, we mean a triple (X, m, e) where X is a topological space and $m:X\times X\to X$ is a map such that $m(x, e)=m(e, x)=x$ for all $x\in X$. Define $m_k:X\to X$ by setting $m_2(x)=m(x, x)$ and, for each $k>2$, $m_k(x)=m(x, m_{k-1}(x))$. The H-space (X, m, e) is *divisible* if m_k is onto for all $k\geqq 2$. If $x, y\in X$ and $m_k(x)=y$, then x is a kth *root* of y. We wish to obtain a result, of the sort Hopf discovered for Lie groups, in the more general setting of H-spaces.

Observe first that we cannot expect to prove that a very large class of H-spaces is divisible. Let I denote the interval $[0, 1]$ and define $m:I\times I\to I$ by $m(s, t)=|s-t|$, then $(I, m, 0)$ is an H-space but $m_2(I)=\{0\}$ so the H-space is not divisible. We will prove, however, that an H-space (X, m, e) is divisible provided that X is a compact connected manifold *without boundary*. This result includes many spaces not covered by Hopf's theorem (see, for example, [**1**] and [**5**]).

Even when we can show that (X, m, e) is divisible, there is no hope for k^λ as a lower bound on the number of kth roots for each $x\in X$, as the following example demonstrates. Let S^3 denote the 3-sphere

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and let $\boldsymbol{H}: S^3 \times S^3 \rightarrow S^3$ be quaternion multiplication. Consider S^3 as the suspension of the 2-sphere, with vertices $e=(1, 0, 0, 0)$ and $(-1, 0, 0, 0)$. Let $\Sigma f: S^3 \rightarrow S^3$ be the suspension of a map $f: S^2 \rightarrow S^2$ of degree 2 then, by the Hopf Homotopy Theorem, there is a homotopy $h_t: S^3 \rightarrow S^3$ such that $h_0 = \boldsymbol{H}_2$ and $h_1 = \Sigma f$. Since S^3 is simply-connected, we may assume that h_t fixes e. Define $\Delta(S^3) = \{(x, x) \mid x \in S^3\}$ and

$$A = S^3 \times \{e\} \cup \{e\} \times S^3 \cup \Delta(S^3) \subset S^3 \times S^3.$$

A map $p: S^3 \times S^3 \times \{0\} \cup A \times I \rightarrow S^3$ is defined by

$$\begin{aligned} p(x, y, t) &= \boldsymbol{H}(x, y) && \text{if} \quad t = 0, \\ &= h_t(x) && \text{if} \quad x = y, \\ &= x && \text{if} \quad y = e, \\ &= y && \text{if} \quad x = e, \end{aligned}$$

then p extends to $P: S^3 \times S^3 \times I \rightarrow S^3$ by the Homotopy Extension Theorem. Define $m: S^3 \times S^3 \rightarrow S^3$ by $m(x, y) = P(x, y, 1)$. Then (S^3, m, e) is an H-space where $m_2 = \Sigma f$ so the only square root of e is e itself even though, in this case, $k^\lambda = 2$. We will prove, however, that if (X, m, e) is an H-space where X is a compact connected manifold without boundary, then each $x \in X$ has at least k^β kth roots, where β denotes the dimension of $H^1(X)$.

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The first thing we shall require is the following computation.

LEMMA. *Let A be a connected Hopf algebra over Λ, a commutative ring with unit, such that A is isomorphic, as an algebra, to the exterior algebra generated by $x_1, \cdots, x_\lambda$. Let φ be the product of A (write $\varphi(x \otimes y) = xy$) and ψ the coproduct. Define $p_2 = \varphi\psi: A \rightarrow A$ and, in general, $p_k: A \rightarrow A$ is defined by $p_k = \varphi(1 \otimes p_{k-1})\psi$ for each integer $k \geqq 3$. Let $\bar{x} = x_1 x_2 \cdots x_\lambda$, then, for all $k \geqq 2$, $p_k(\bar{x}) = k^\lambda \bar{x}$.*

PROOF. Since $A = \sum A_p$ is a graded Λ-module, for $x \in A_p$ we define the *degree* of x by $\deg(x) = p$. Order the generators $x_1, \cdots, x_\lambda$ so that $\deg(x_i) \leqq \deg(x_{i+1})$ for all $i = 1, \cdots, \lambda - 1$. The algebra A is generated as a Λ-module by all monomials $y_j = x_{j(1)} \cdots x_{j(r)}$, where $1 \leqq j(1) < \cdots < j(r) \leqq \lambda$. Define the *weight* $w(y_j)$ of the monomial y_j by $w(y_j) = \deg(x_{j(r)})$. By definition, $p_2 = \varphi\psi$ so, for each $i = 1, \cdots, \lambda$,

$$\begin{aligned} p_2(x_i) &= \varphi\psi(x_i) \\ &= \varphi(x_i \otimes 1 + 1 \otimes x_i + \sum a_j y_j \otimes y_j') \\ &= 2x_i + \sum a_j y_j y_j' \end{aligned}$$

where $a_j \in \Lambda$, $w(y_j) < \deg(x_i)$ and $w(y_j') < \deg(x_i)$. Since A is an exterior algebra, either $y_j y_j' = 0$, because y_j and y_j' have a generator in common, or $y_j y_j'$ is again a generating monomial (up to sign) and since

$$w(y_j y_j') = \max\{w(y_j), w(y_j')\}$$

then $w(y_j y_j') < \deg(x_i)$. Thus we may write, for $i = 1, \cdots, \lambda$,

$$p_2(x_i) = 2x_i + \sum a_j'' y_j'$$

where $a_j'' \in \Lambda$ and $w(y_j'') < \deg(x_i)$. Suppose that, for $i = 1, \cdots, \lambda$,

$$p_{k-1}(x_i) = (k-1)x_i + \sum a_j' \, y_j'$$

where $a_j'' \in \Lambda$ and $w(y_j'') < \deg(x_i)$, then

$$\begin{aligned} p_k(x_i) &= \varphi(1 \otimes p_{k-1})\psi(x_i) \\ &= \varphi(1 \otimes p_{k-1})(x_i \otimes 1 + 1 \otimes x_i + \sum a_j y_j \otimes y_j') \\ &= x_i + p_{k-1}(x_i) + \sum a_j y_j (p_{k-1}(y_j')) \\ &= kx_i + \sum a_j'' y_j'' + \sum a_j y_j (p_{k-1}(y_j')). \end{aligned}$$

Let $y_j' = x_{j(1)} \cdots x_{j(r)}$, then

$$p_{k-1}(y_j') = p_{k-1}(x_{j(1)}) \cdots p_{k-1}(x_{j(r)}).$$

Of course $p_{k-1}(x_{j(q)}) \in A_{j(q)}$ so $p_{k-1}(x_{j(q)})$ is a linear combination of monomials of weight no greater than $j(q)$, which is less than $\deg(x_i)$. Therefore $p_{k-1}(y_j')$ is a linear combination of monomials of weight less than $\deg(x_i)$. Since the monomials y_j are of weight less than $\deg(x_i)$ and, by the induction hypothesis, the same is true of the y_j'', we have proved, for all integers $k \geqq 2$ and each generator x_i, $i = 1, \cdots, \lambda$, that

$$p_k(x_i) = kx_i + \sum a_j'' y_j''$$

where $a_j'' \in \Lambda$ and $w(y_j'') < \deg(x_i)$. Obviously $p_k(x_1) = kx_1$ because there are no generators of lower degree. Suppose, for some $\mu < \lambda$, that

$$p_k(x_1) \cdots p_k(x_\mu) = k^\mu (x_1 \cdots x_\mu)$$

then

$$\begin{aligned} p_k(x_1) \cdots p_k(x_{\mu+1}) &= k^\mu (x_1 \cdots x_\mu) p_k(x_{\mu+1}) \\ &= k^{\mu+1}(x_1 \cdots x_{\mu+1}) + k^\mu (x_1 \cdots x_\mu) \sum a_j' \, y_j'' \\ &= k^{\mu+1}(x_1 \cdots x_{\mu+1}) + k^\mu \sum a_j'' (x_1 \cdots x_\mu) y_j' \; . \end{aligned}$$

But $w(y_j'') < \deg(x_{\mu+1})$ so $y_j'' = x_{j(1)} \cdots x_{j(r)}$ where $j(q) \leqq \mu$ for all $q = 1, \cdots, r$ because of the order imposed on the x_i. Therefore, since

A is an exterior algebra, $(x_1 \cdots x_\mu)y_j'' = 0$ and we have

$$p_k(x_1) \cdots p_k(x_{\mu+1}) = k^{\mu+1}(x_1 \cdots x_{\mu+1}).$$

Thus, in a finite number of steps, we obtain

$$p_k(\bar{x}) = p_k(x_1) \cdots p_k(x_\lambda) = k^\lambda(x_1 \cdots x_\lambda) = k^\lambda \bar{x}.$$

By an *H-manifold*, we shall mean an H-space (M, m, e) where M is a compact connected manifold without boundary.

Let (M, m, e) be an H-manifold and let $x_1, \cdots, x_\lambda$ generate $H^*(M)$, then since M is orientable [3], $\bar{x} = x_1x_2 \cdots x_\lambda \in H^n(M)$, where n is the dimension of M. Define $\Delta: M \to M \times M$ to be the diagonal map. Then $H^*(M)$ is a connected Hopf algebra over the rationals with product Δ^* and coproduct m^*. Observe that we defined $m_2 = m\Delta$ and the maps m_k for $k > 2$ so that the diagram

$$\begin{tikzcd} M \arrow[r, "m_k"] \arrow[d, "\Delta"'] & M \\ M \times M \arrow[r, "1 \times m_{k-1}"'] & M \times M \arrow[u, "m"'] \end{tikzcd}$$

commutes. Therefore, by the lemma,

$$\overset{*}{m_k}(\bar{x}) = p_k(\bar{x}) = k^\lambda \bar{x} \neq 0,$$

for all $k \geqq 2$.

By [2], since $m_k^*: H^n(M) \to H^n(M)$ is not the zero homomorphism then, for each $x \in M$, there is a lower bound for the number of kth roots of x, namely, the order of the cokernel of the induced homomorphism

$$m_{k*}: \pi_1(M, e) \to \pi_1(M, e).$$

It is well known that $m_{k*}(\alpha) = k\alpha$ for all $\alpha \in \pi_1(M, e)$.

By the Fundamental Theorem of Abelian Groups, we write

$$\pi_1(M, e) \cong Z^{(1)} \oplus \cdots \oplus Z^{(\beta)} \oplus T = F \oplus T$$

where each $Z^{(i)}$ is infinite cyclic and T is finite. By the Hurewicz Isomorphism Theorem and the Universal Coefficient Theorem, β is the dimension of $H^1(M)$. Let m'_{k*} denote the restriction of m_{k*} to F, then the cokernel of m'_{k*} is $Z_k^{(1)} \oplus \cdots \oplus Z_k^{(\beta)}$ where $Z_k^{(i)}$ is a cyclic group of order k. Therefore, the cokernel of m'_{k*} has order k^β and since the order of the cokernel of m_{k*} is at least as large, we have proved

THEOREM. *Every H-manifold (M, m, e) is divisible. Moreover, each $x \in M$ has at least k^β kth roots for all integers $k \geqq 2$, where β denotes the dimension of $H^1(M)$.*

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