ON ARHANGEL'SKIĬ'S CLASS MOBI

H. R. BENNETT

1. **Introduction.** In [1], A. V. Arhangel'skiĭ defines the class MOBI and asks whether each topological space in MOBI is either developable, pointwise paracompact[1] or a p-space.[2] It is further asserted that each topological space in MOBI has closed sets G_δ[3] and asked if a Lindelöf space or a paracompact space in MOBI is metrizable. These questions are answered negatively.

The notation will follow [4] and all spaces will be at least Hausdorff.

2. **The class** MOBI.

DEFINITION 2.1. The class MOBI is the intersection of all classes of topological spaces satisfying the two conditions:

(i) Every metric space belongs to the class.

(ii) The image of a space in the class under a compact open map[4] is also in the class.

THEOREM 2.2. *A topological space y is in* MOBI *if and only if there is a metric space M and a finite set* $\{\phi_1, \cdots, \phi_n\}$ *of compact open maps such that* $(\phi_n \circ \cdots \circ \phi_1)(M) = Y$.

PROOF. Let $\mathrm{MOBI} = \bigcap\{C_\alpha \mid \alpha \in A\}$ and let $B = \{X \mid X$ is a metric space$\} \cup \{Y \mid$ there is a metric space M and a finite set $\{\phi_1, \cdots, \phi_n\}$ of compact open maps such that $(\phi_n \circ \cdots \circ \phi_1)(M) = Y\}$.

There is an α in A such that $B = C_\alpha$ since if Y is in B, then either Y is a metric space and, consequently, in C_α or $Y = (\phi_n \circ \cdots \circ \phi_1)(M)$ where M is metric and each ϕ_i, $1 \leqq i \leqq n$, are compact open maps. Thus if ϕ is a compact open map $\phi(Y) = \phi((\phi_n \circ \cdots \circ \phi_1)(M))$ is, by definition, in B. Thus $B = C_\alpha \supseteq \mathrm{MOBI}$.

Received by the editors June 9, 1969.

AMS 1969 *subject classifications*. Primary 5460; Secondary 5438, 5450.

[1] A space is pointwise paracompact (=weakly paracompact in Russian) if each open cover has a point finite open refinement.

[2] A topological space S is a p-space provided that for some Hausdorff compactification T of S there is a sequence $P_1, P_2, \cdots$ of collections of open subsets of T, each of which covers S, such that if x is any element of S, then $\bigcap_{i=1}^{\infty} \{\bigcup\{p \in P_i \mid x \in p\}\}$ is contained in S.

[3] A set A is a G_δ if it is the countable intersection of open sets.

[4] A map is a continuous function. A map is compact open if the image of an open set is open and if the inverse image of a point is compact.

For each α in A, $B \subseteq C_\alpha$. For if Y is in B, then either Y is metric and in C_α or $Y = (\phi_n \circ \cdots \circ \phi_1)(M)$ where M is metric and the ϕ_i, $1 \leqq i \leqq n$, are compact open maps. Since M is in C_α, $\phi_1(M)$ is also in C_α and inductively $(\phi_n \circ \cdots \circ \phi_1)(M) = Y$ is in C_α. Thus $B \subseteq C_\alpha$ for each α. It follows that $B = \bigcap \{ C_\alpha \mid \alpha \in A \} = \text{MOBI}$.

COROLLARY 2.3. *Each pointwise paracompact, developable Hausdorff space is in* MOBI.

PROOF. Theorem 2.2 and Theorem 5 of [**5**].

DEFINITION 2.4. A sequence $G_1, G_2, \cdots$ of collections of open subsets of a topological space X is called a quasi-development for X provided that for each point p of X and each open set R containing p there is a natural number n such that p belongs to some element of G_n and each element of G_n that contains p lies in R. If, in addition, each G_n is a cover of X, then $G_1, G_2, \cdots$ is a development. A space having a (quasi-) development (see [**3**]) is said to be (quasi-) developable.

THEOREM 2.5. *There exists a regular, Lindelöf, hereditarily paracompact, quasi-developable space* Y[5] *in* MOBI *that is neither developable, a p-space, nor has closed sets* G_δ.

PROOF. Let Q be an uncountable subset of $[0,1]$ whose only compact subsets are countable; such spaces exist [**7**, p. 422]. Let $T = [0, 1] - Q$ and if N denotes the natural numbers let $X = \{(x, y) \mid 0 \leqq x \leqq 1,\ y = 0 \text{ or } x \in Q,\ y \in N\}$. Topologize X by letting a set be open if it is the union of any of the following types of sets:

(i) $\{(x, y)\}$ if $0 \leqq x \leqq 1$ and $y > 0$,

(ii) $\{(x, 0)\} \cup \{(x, y) \mid y > c\}$ if $x \in Q$ and $c > 0$,

(iii) $\{(x, 0) \mid x \in T,\ a < x < b\} \cup \{(x, y) \mid a < x < b,\ x \in Q,\ y > c\}$ where $a < b$ and $c > 0$.

It follows that X is a Hausdorff, pointwise paracompact, developable space. Therefore, by Corollary 2.3, X is a MOBI.

Let Y be $[0, 1]$ retopologized to make the points of Q discrete. As noted in [**6**], Y is hereditarily paracompact and Lindelöf. Clearly $Y - Q$ is closed and not a G_δ in Y; thus Y is not developable. If $\{B_1, B_2, \cdots\}$ is a countable base for $[0, 1]$, by letting $G_1 = \{\{x\} \mid x \in Q\}$ and, for $i > 1$, $G_i = \{B_{i-1}\}$, it follows that $G_1, G_2, \cdots$ is a quasi-development for Y. It is shown in [**6**] that the product of Y with a metric space (the irrationals) is not normal. Thus, by a result of A. V. Arhangel'skiĭ (Corollary 1 of Theorem 16 in [**2**]) it follows that Y is not a p-space.

---

[5] The space in Example 2 of [**8**] is similar to the space Y.

Let a map ϕ be defined by $\phi((x, y)) = x$. It follows that ϕ is an open compact map with domain X and range Y. By Theorem 2.2 Y is in MOBI.

Thus most of the questions in the group of problems 5.7 in [1] are answered negatively. Example 3 of [8] shows that each element of MOBI need not be pointwise paracompact.

QUESTION 1. Is each member of MOBI a quasi-developable space?

QUESTION 2.[6] If the members of MOBI are restricted to completely regular spaces, can the questions in 5.7 of [1] still be answered negatively?

BIBLIOGRAPHY

1. A. V. Arhangel'skiĭ, *Mappings and spaces*, Uspehi Mat. Nauk **21** (1966), no. 4 (130), 133–184 = Russian Math. Surveys **21** (1966), no. 4, 115–162. MR **37** #3534.

2. ———, *On a class of spaces containing all metric and all locally compact spaces*, Dokl. Akad. Nauk SSSR **151** (1963), 751–754 = Soviet Math. Dokl. **4** (1963), 1051–1055. MR **27** #2959.

3. H. R. Bennett, *Quasi-developable spaces*, Dissertation, Arizona State University, Tempe, Arizona, 1968.

4. J. Dugundi, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR **33** #1824.

5. Sitiro Hanai, *On open mappings*. II, Proc. Japan Acad. **37** (1961), 233–238. MR **23** #A4116.

6. E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. **69** (1963), 375–376. MR **27** #2956.

7. C. Kuratowski, *Topologie*. Vol. I, PWN, Warsaw, 1958; English transl., Academic Press, New York and PWN, Warsaw, 1966. MR **19**, 873; MR **36** #839; 840.

8. H. H. Wicke and J. M. Worrell, *Open continuous mappings of spaces having bases of countable order*, Duke Math. J. **34** (1967), 255–271. MR **35** #979.

TEXAS TECHNOLOGICAL UNIVERSITY, LUBBOCK, TEXAS 79406

---

[6] This question is due to the referee, whom the author would like to thank for his valuable suggestions.