TREE-LIKE CONTINUA AND CELLULARITY

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ABSTRACT. In this paper the equivalence of tree-like and cellular is proved for 1-dimensional continua in E^n. More precisely, if X is a tree-like continuum, then the collection of all embeddings $h: X \to E^n$, $n \geqq 3$, such that $h[X]$ is cellular in E^n is a dense G_δ-subset of the collection of all maps from X into E^n. Conversely, if X is a 1-dimensional cellular subset of E^n, then X is a tree-like continuum.

1. **Terminology.** Throughout this paper a continuum will be a nondegenerate compact connected metric space and a covering will be a finite open covering. The symbol $\sim$ should be translated "homotopic to." If X is a continuum and $\mathcal{O} = \{O_1, \cdots, O_m\}$ is a covering of X, the *mesh* of $\mathcal{O}$, denoted mesh $\mathcal{O}$, is the maximum of the diameters of the elements of $\mathcal{O}$. The *nerve* of $\mathcal{O}$, denoted $\mathfrak{N}(\mathcal{O})$, is the abstract complex consisting of those simplexes $(O_{i_1} \cdots O_{i_j})$ such that $O_{i_1} \cap \cdots \cap O_{i_j} \neq \varnothing$. A continuum X is *tree-like* if for each $\epsilon > 0$ there exists a covering $\mathcal{O}$ of X such that mesh $\mathcal{O} < \epsilon$ and $\mathfrak{N}(\mathcal{O})$ is a contractible 1-complex.

Let X be a subset of a topological space Y and let n be a nonnegative integer. The statement that X has *property n-UV* means that for each open set U containing X, there is an open set V containing X and contained in U such that each singular n-sphere in V is homotopic to 0 in U. X has *property UV^n* if it has property i-UV for each $i \leqq n$ and X has *property UV^ω* if it has property i-UV for each nonnegative integer i. X has *property UV^∞* if for each open set U containing X, there is an open set V containing X and contained in U such that V is contractible in U. For a good discussion of the UV properties the reader is referred to Armentrout [1].

A subset X of E^n is said to be *cellular* in E^n if there is a sequence $C_1, C_2, \cdots$ of n-cells in E^n such that

(1) for each positive integer i, $C_{i+1} \subset \operatorname{Int} C_i$, and

(2) $\bigcap_{i=1}^\infty C_i = X$.

This paper is devoted to studying the relationship between tree-like, the UV-properties, and cellularity in Euclidean space. In §2 we show that for 1-dimensional continua they are essentially the same and in §3 we prove an embedding theorem for tree-like continua.

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2. **An equivalence theorem.** In this section we shall show that a 1-dimensional continuum X is tree-like if and only if the image of each embedding of X into E^n has property UV^∞. This is equivalent to the statement that there is an embedding h of X into some Euclidean space such that $h[X]$ is cellular.

LEMMA 1. *Let X be a continuum in E^n, $n \geqq 3$. If X is 1-dimensional, then X has property i-UV for $i=0, 2, 3, \cdots$. If X is tree-like, then X has property UV^ω.*

PROOF. Let U and W be open subsets of E^n such that $\overline{W}$ is compact and $X \subset W \subset \overline{W} \subset U$. There is a positive real number ϵ such that if A is any subset of U which meets $\overline{W}$ and has diameter less than ϵ, then the convex hull of A is contained in U.

Let $\mathcal{O} = \{O_1, \cdots, O_m\}$ be a covering of X by open sets contained in W such that mesh $\mathcal{O} < \epsilon/3$ and $\mathfrak{N}(\mathcal{O})$ is a 1-complex. If X is tree-like then $\mathcal{O}$ may be chosen so that $\mathfrak{N}(\mathcal{O})$ is contractible. For each $i=1, \cdots, m$, let p_i be a point of O_i such that the set $\{p_1, \cdots, p_m\}$ is in general position in E^n. Since $n \geqq 3$, the collection L consisting of vertices $p_1, \cdots, p_m$ and 1-simplexes (p_ip_j) such that $O_i \cap O_j \neq \varnothing$ is a subcomplex of E^n. The choice of ϵ implies that L is contained in U. Moreover, L is the image under a simplicial embedding of $\mathfrak{N}(\mathcal{O})$ into E^n and therefore is contractible if X is tree-like.

Let $V = \bigcup_{i=1}^m O_i$. Using the methods employed in [4, p. 69], there is a mapping f from V onto L such that $O_i = f^{-1}[\text{s}^\circ\text{t } p_i]$ (here $\text{s}^\circ\text{t } p_i$ denotes the open star of p_i in L). Note that f moves no point x in V more than ϵ, for if $x \in O_i$, then $d(x, f(x)) \leqq d(x, p_i) + d(p_i, f(x)) < \epsilon/3 + 2\epsilon/3 = \epsilon$.

Now let S denote the standard k-dimensional sphere for some nonnegative integer k and let $g: S \to V$ be a map. Then fg maps S into $L \subset U$ and $d(g(y), fg(y)) < \epsilon$ for each $y \in S$. Thus fg and g are homotopic in E^n by a homotopy which moves $fg(y)$ to $g(y)$ along a straight line segment of length less than ϵ. In particular, the choice of ϵ implies that fg and g are homotopic in U. But $fg[S]$ is contained in L and therefore, if $k \neq 1$, $fg \sim 0$ in $L \subset U$. If X is tree-like, then $fg \sim 0$ in $L \subset U$ for all nonnegative integers k. Thus $g \sim fg \sim 0$ in U for the desired cases.

The next lemma is proved by Case and Chamberlin in [2].

LEMMA 2. *A 1-dimensional continuum is tree-like if and only if each continuous map of X into any linear graph is homotopic to 0.*

LEMMA 3. *If X is a 1-dimensional continuum in E^n having property UV^∞, then X is tree-like.*

PROOF. Let $g:X\to K$ be a map from X into a linear graph K. Since g is homotopic to a map from X onto a subcomplex of K, there is no loss of generality in assuming that g is onto. Let $p_1, \cdots, p_m$ be the vertices of K and for each $i=1, \cdots, m$, let $O_i=g^{-1}[\text{s}^\circ\text{t}\ p_i]$. Then $\mathcal{O}=\{O_i\}$ is a covering of X and $\mathfrak{N}(\mathcal{O})$ is a 1-complex simplicially isomorphic to K. Let $U_1, \cdots, U_m$ be open subsets of E^n such that $U_i\cap X=O_i$ for $i=1, \cdots, m$ and such that if $\mathfrak{U}=\{U_i\}$, then $\mathfrak{N}(\mathfrak{U})$ is simplicially isomorphic to $\mathfrak{N}(\mathcal{O})$. Let $U=\bigcup_{i=1}^m U_i$ and let $f: U\to K$ be a map such that $f^{-1}[\text{s}^\circ\text{t}\ p_i]=U_i$ for $i=1, \cdots, m$. Note that for each $x\in X$, $f(x)$ and $g(x)$ lie in the same simplex of K and therefore $g\sim f|X$ in K. We show $f|X$ is homotopic to 0 in K.

Now X has property UV^∞ in E^n and U is an open set containing X, so there is a homotopy $H':X\times[0, 1]\to U$ such that $H'(x, 0)=x$ and $H'(x, 1)=x_0$ for some point $x_0\in U$. Define $H:X\times[0, 1]\to K$ by $H=f\circ H'$. Then $H(x, 0)=f(x)$ and $H(x, 1)=f(x_0)$.

THEOREM 1. *If X is a 1-dimensional continuum, then the following are equivalent*:

(1) *X is tree-like,*

(2) *the image of each embedding of X into E^n has property 1-UV,*

(3) *the image of each embedding of X into E^n has property UV^∞, and*

(4) *X can be embedded as a cellular subset of some Euclidean space.*

PROOF. If $n\geqq 3$, then the implications (1)⇒(2)⇒(3) follow directly from Lemma 1. If $n<3$, then Lemma 5.1 of [1] applies.

If $h:X\to E^n$ is an embedding such that $h[X]$ has property UV^∞, then McMillan [5] has shown that $h[X]$ is cellular in E^{n+1}. Thus (3) and (4) are equivalent (observing Lemma 5.1 of [1] again). The proof is then completed by applying Lemma 3.

The previous theorem and the results of [2] provide an interesting example concerning the UV-properties. Case and Chamberlin construct an example of a subset X of E^3 which is not tree-like, but which has trivial Čech groups.

COROLLARY 1. *There is a 1-dimensional continuum X in E^3 which has trivial Čech homology groups, cohomology groups, and fundamental group, but not having property UV^∞ in E^3.*

2. **Embeddings of tree-like continua in E^n.** Throughout this section let X be a fixed tree-like continuum. Let $F[X]$ denote the collection of all mappings from X into E^n with the compact open topology. Recall that $F[X]$ is a complete metric space (cf. [4]) with the usual sup metric.

Consider the following subsets of $F[X]$:

$$I[X] = \{f \in F[X] \mid f \text{ is an embedding}\},$$
$$F_c[X] = \{f \in F[X] \mid f[X] \text{ is cellular in } E^n\},$$
$$I_c[X] = F_c[X] \cap I[X].$$

In this section we prove that if $n \geqq 3$, then $I_c[X]$ is a dense G_δ-subset of $F[X]$. Note that if $n<3$, then $I_c[X]=I[X]$. We assume therefore that n is a fixed integer $\geqq 3$.

If ϵ is a positive real number, an *ϵ-mapping* $f:X \rightarrow E^n$ is an element of $F[X]$ such that for each $y \in f[X]$, the set $f^{-1}(y)$ has diameter less than ϵ. For each $i=1, 2, \cdots$, let G_i be the subset of $F[X]$ consisting of all $1/i$-mappings. The following result is proved in [4].

LEMMA 4. *For each positive integer i, G_i is a dense open subset of $F[X]$. Moreover, $I[X]=\bigcap_{i=1}^{\infty} G_i$ is a dense G_δ-subset of $F[X]$.*

For each $i=1, 2, \cdots$, let C_i be the collection of all elements f of $F[X]$ such that there is an n-cell C in E^n with $f[X] \subset \operatorname{Int} C \subset C \subset N(f[X], 1/i)$. (Here, the set $N(f[X], 1/i)$ denotes the $1/i$-neighborhood of $f[X]$ in E^n.) Clearly $F_c[X]=\bigcap_{i=1}^{\infty} C_i$. The following two lemmas show that each C_i is a dense open subset of $F[X]$.

LEMMA 5. *$F_c[X]$ is dense in $F[X]$.*

PROOF. Let g be an element of $F[X]$ and let ϵ be a positive real number. Lemma 4 implies that X can be considered a subset of E^n such that g moves no point more than $\epsilon/2$. Corresponding to $\epsilon/2$, let f and L be as in the proof of Lemma 1; that is, f maps X onto the contractible 1-complex L in E^n without moving points more than $\epsilon/2$. Then f and g are within ϵ of each other and, since L is collapsible, $f[X]$ is cellular in E^n.

LEMMA 6. *For each positive integer i, C_i is an open subset of $F[X]$.*

PROOF. Suppose $f \in C_i$ and let C be an n-cell in E^n such that $f[X] \subset \operatorname{Int} C \subset C \subset N(f[X], 1/i)$. Let

$$\epsilon = \min\{d(f[X], E^n - \operatorname{Int} C), d(C, E^n - N(f[X], 1/i))\}.$$

Since $\epsilon < 1/i$, any $\epsilon/2$-approximation g to f will have the property that $g[X] \subset \operatorname{Int} C \subset C \subset N(g[X], 1/i)$.

THEOREM 2. *If $n \geqq 3$, $I_c[X]$ is a dense G_δ-subset of $F[X]$.*

PROOF. The previous lemmas imply that for each $i=1, 2, \cdots$, both G_i and C_i are dense and open in $F[X]$. Thus $G_i \cap C_i$ is dense and open. By Theorem 2–79 of [3], the set $\bigcap_{i=1}^{\infty} (G_i \cap C_i) = I[X] \cap F_c[X] = I_c[X]$ is a dense G_δ-subset of $F[X]$.

The following corollary is now obvious.

COROLLARY 2. *Let X be a 1-dimension continuum in E^n having property UV^∞ and let ϵ be a positive real number. Then there is an embedding $h: X \to E^n$ such that $d(x, h(x)) < \epsilon$ for each $x \in X$ and such that $h[X]$ is cellular in E^n.*

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