LINEAR PERTURBATIONS OF ORDINARY DIFFERENTIAL EQUATIONS[1]

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ABSTRACT. We present several results dealing with the problem of the preservation of the stability of a system $x' = A(t)x$ which is subject to linear perturbations $B(t)x$, or to perturbations dominated by linear ones.

1. We present several results dealing with the problem of the preservation of the stability and continuous dependence for a system $x' = A(t)x$ to which is added a linear perturbation $B(t)x$, or a perturbation dominated by a linear one. In §2 we show that certain linear perturbations satisfying $\int_0^\infty |B(t)| dt < \infty$, which have little effect on an exponentially stable linear system, have a rather significant effect on linear systems possessing a slightly weaker stability property. In §3 we consider $B(t) \equiv B$, a constant matrix, and we answer the following question: if $x = 0$ is to be exponentially stable for $x' = A(t)x + Bx$ no matter which exponentially stable system $x' = A(t)x$ is considered, then what special form must B have? In §4 we give a new proof and slight extension of the following known result [4]: a necessary and sufficient condition that a fundamental matrix of $x' = A(t)x + R_n(t)x$ converge is that a fundamental matrix of $y' = R_n(t)y$ converge as $n \to \infty$.

DEFINITION. We say that a linear system

$$(L) \qquad x' = A(t)x$$

is *exponentially stable* if there exist constants $K \geqq 1$ and $\sigma > 0$ such that

$$|X(t)X^{-1}(s)| \leqq Ke^{-\sigma(t-s)} \qquad \text{for all } t \geqq s \geqq 0,$$

where $X(t)$ denotes a fundamental matrix of (L).

Unless otherwise stated, all functions $f(t, x)$ considered here are required to be continuous for all $t \geqq 0$ and x in R^d, $d \geqq 1$.

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2. The following result is known.

THEOREM 1. *Let* (L) *be exponentially stable, with corresponding constants K and σ. Let $g(t, x)$ satisfy any one of the following three conditions for all $t \geqq 0$ and x in R^d.*

(2.1) $$|g(t, x)| \leqq \gamma |x|, \quad \textit{where} \quad \gamma < \sigma/K.$$

(2.2) $$|g(t, x)| \leqq \gamma(t) |x|, \quad \textit{where} \quad t^{-1}\int_0^t \gamma(s)ds \to 0 \textit{ as } t \to \infty.$$

(2.3) $$|g(t, x)| \leqq \gamma(t), \quad \textit{where} \quad \int_t^{t+1} \gamma(s)ds \to 0 \textit{ as } t \to \infty.$$

Then all solutions of $x' = A(t)x + g(t, x)$ approach zero as $t \to \infty$.

Stronger conclusions are also known, but this one suffices for our purpose. A proof using (2.1) may be found in [1], one using (2.2) in [3], and one using (2.3) in [5]. W. A. Coppel [2] has shown that the part using (2.1) is still true if one replaces the exponential stability of (L) by the weaker assumption that, for some $K \geqq 1$ and $\sigma > 0$,

(2.4) $$\sup_{t \geqq 0} \int_0^t |X(t)X^{-1}(s)| \, ds \leqq K/\sigma.$$

The purpose of this section is to show that the parts using (2.2) and (2.3) do not follow under the assumption (2.4), even when $\int_0^\infty \gamma(s)ds < \infty$.

EXAMPLE 1. We construct a two-dimensional linear system (L) for which (2.4) holds, and we define a matrix $B(t)$ satisfying $\int_0^\infty |B(t)| dt < \infty$, such that "most" solutions of $x' = A(t)x + B(t)x$ are unbounded as $t \to \infty$. Define the intervals

$$I_n = [n - 2^{-(3n+1)}, n + 2^{-(3n+1)}], \qquad J_n = [n - 2^{-3n}, n + 2^{-3n}],$$

for $n = 1, 2, \cdots$. Define two C^∞ functions $\lambda(t)$ and $\phi(t)$ on $[0, \infty)$ such that

$$\begin{aligned} \lambda(t) &= 2^{2n} \quad \text{if } t \in I_n, & \phi(t) &= 2^{2n} \quad \text{if } t \in I_n, \\ &= 1 \quad \text{if } t \notin J_n, & &= 0 \quad \text{if } t \notin J_n, \end{aligned}$$

and so that $\lambda(t)$ and $\phi(t)$ are each monotone in each component of $J_n - I_n$, $n = 1, 2, \cdots$. Thus

$$\int_{J_n} \lambda(t)dt \leqq 2^{-n+1}, \qquad \int_{J_n} \phi(t)dt \leqq 2^{-n+1}, \qquad \int_{J_n} \phi(t)\lambda(t)dt \geqq 2^n.$$

Now define

$$A(t) = \begin{pmatrix} -1 - \lambda'(t)\lambda^{-1}(t) & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 0 & \phi(t) \\ 0 & 0 \end{pmatrix}.$$

Therefore $\int_0^\infty |B(t)|dt < \infty$. We now show that (L) satisfies (2.4). A fundamental matrix is

$$X(t) = \begin{pmatrix} \lambda^{-1}(t)e^{-t} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Since

$$\int_0^t e^{-(t-s)}\lambda^{-1}(t)\lambda(s)ds \leqq \int_0^t e^{-(t-s)}ds + \sum_{n=1}^{[t+1]} \int_{J_n} \lambda(s)ds$$
$$\leqq 1 + \sum_{n=1}^{\infty} 2^{-n+1},$$

(2.4) holds for (L) with $K=6$, $\sigma=\frac{1}{2}$. However, there is a solution $y(t) = (y_1(t), y_2(t))$ of $y' = A(t)y + B(t)y$ satisfying

$$y_1(t) = \lambda^{-1}(t)e^{-t}\int_0^t e^s\lambda(s)\phi(s)e^{-s/2}ds, \qquad y_2(t) = e^{-t/2}.$$

Hence

$$y_1(n + 2^{-3n}) = e^{-n-2^{-3n}}\int_0^{n+2^{-3n}} e^{s/2}\lambda(s)\phi(s)ds$$
$$\geqq e^{-n-2^{-3n}}e^{(n-2^{-3n})/2}\int_{J_n} \lambda(s)\phi(s)ds$$
$$\geqq \left(\frac{2}{e^{1/2}}\right)^n e^{-2^{-3n}-2^{-3n-1}} \to \infty \quad \text{as } n \to \infty.$$

EXAMPLE 2. We construct a one-dimensional linear system (L) for which (2.4) holds, and we define a function $\phi(t) \geqq 0$ satisfying $\int_0^\infty \phi(t)dt < \infty$, such that all solutions of $x' = A(t)x + \phi(t)$ are unbounded as $t \to \infty$. Define I_n, J_n, $\lambda(t)$, and $\phi(t)$ as in Example 1. Then the linear (one-dimensional) system

$$x' = (-1 - \lambda'(t)\lambda^{-1}(t))x \tag{2.5}$$

satisfies (2.4). Also $\int_0^\infty \phi(t)dt < \infty$. But there is a solution $z(t)$ of

$$z' = (-1 - \lambda'(t)\lambda^{-1}(t))z + \phi(t) \tag{2.6}$$

satisfying

$$z(t) = \lambda^{-1}(t)e^{-t}\int_0^t e^s\lambda(s)\phi(s)ds \geqq y_1(t),$$

where $y_1(t)$ is as in Example 1. Thus $z(t)$ is unbounded as $t\to\infty$. Since every solution of (2.5) is bounded as $t\to\infty$, it follows that all solutions of (2.6) are unbounded as $t\to\infty$.

Notice that in each of the above two examples, the perturbation is unbounded as $t\to\infty$. The problem of what happens if it is assumed to be also bounded remains open.

3. Let $\mathcal{A}$ be the family of all $d\times d$ matrices $A(t)$, continuous on $[0, \infty)$, for which (L) is exponentially stable. We say that a matrix B *perturbs* $\mathcal{A}$ if $x' = A(t)x + Bx$ is exponentially stable for every $A(t)\in\mathcal{A}$. This problem arises in the following way: suppose one knows that a system is both linear and exponentially stable. Suppose *no further information about the system can be obtained.* If this system is subject to linear perturbations Bx, one would like to know that stability is preserved no matter what linear, exponentially stable unperturbed system (L) was at hand. This is the motivation behind asking that Bx perturb a whole class of equations (L). If I denotes the identity matrix, then it is obvious that αI perturbs $\mathcal{A}$ if and only if $\alpha\leqq 0$. It is perhaps surprising that even if B is diagonal with all diagonal entries nonpositive, then B does not perturb $\mathcal{A}$ unless all the diagonal entries are equal. More generally, we have the following result.

THEOREM 2. *B perturbs $\mathcal{A}$ if and only if $B = \alpha I$ for some $\alpha\leqq 0$.*

PROOF. Let $B = \alpha I$ for some $\alpha \leqq 0$. Let $X(t)$ and $Y(t)$ denote fundamental matrices of (L) and

$$y' = A(t)y + By, \tag{3.1}$$

respectively, $X(0) = Y(0) = I$. Then $Y(t) = e^{\alpha t}X(t)$. Thus B perturbs $\mathcal{A}$.

Conversely, let $B\neq\alpha I$ for all $\alpha\leqq 0$. If $B = \alpha I$ for some $\alpha > 0$, then B clearly does not perturb $\mathcal{A}$ (choose $A = -\frac{1}{2}B$). Thus suppose that $B\neq\alpha I$ for all real α. Then $d\geqq 2$ and there exists $x_1\in R^d$ such that x_1 and Bx_1 are linearly independent. Define $x_2 = -x_1 + Bx_1$. Choose $x_3, \cdots, x_d$ so that $\{x_1, \cdots, x_d\}$ is a basis. Define A such that $Ax_1 = -x_1 - x_2$, $Ax_2 = x_1 - x_2$, and $Ax_i = -x_i$ for $i = 3, \cdots, d$. Then $A\in\mathcal{A}$ because

$$e^{-t}(x_1\cos t + x_2\sin t), \qquad e^{-t}(x_1\sin t - x_2\cos t), \qquad x_3e^{-t}, \cdots, x_de^{-t},$$

form a set of d linearly independent solutions to $x' = Ax$. But (3.1) is not exponentially stable because $Ax_1+Bx_1=0$. Thus B does not perturb $\mathfrak{A}$.

REMARK. In the "only if" part of the above proof, the matrix $A(t)$ is constant. Thus Theorem 2 holds with $\mathfrak{A}$ replaced by $\mathfrak{A}_c$, the family of all constant $d\times d$ matrices for which (L) is exponentially stable.

4. The purpose of this section is to give a new proof of the following result due to A. Ju. Levin [4], who proved the result in the case where $f(t, x)=A(t)x$.

THEOREM 3. *Let* $x_n(t)$, $Y_n(t)$, *and* $x(t)$ *satisfy*

$$x' = f(t, x) + R_n(t)x, \qquad x(0) = x_0, \tag{4.1}$$

$$Y' = R_n(t)Y, \qquad Y(0) = I, \tag{4.2}$$

$$x' = f(t, x), \qquad x(0) = x_0, \tag{4.3}$$

respectively. Assume that solutions of (4.3) *are uniquely determined by* x_0. *Let* $T>0$. *Then as* $n\to\infty$

$$Y_n(t) \to I \quad \textit{uniformly on} \quad [0, T] \tag{4.4}$$

implies

$$x_n(t) \to x(t) \quad \textit{uniformly on} \quad [0, T] \quad \textit{for every} \quad x_0 \in R^d. \tag{4.5}$$

If $f(t, x)=A(t)x$, *then* (4.5) *implies* (4.4).

PROOF. Define $w_n(t)=Y_n^{-1}(t)x_n(t)$. Then $w_n(t)$ satisfies

$$w' = Y_n^{-1}(t)f(t, Y_n(t)w), \qquad w(0) = x_0. \tag{4.6}$$

Suppose $Y_n(t)\to I$ uniformly. Since the right-hand side of (4.6) converges to that of (4.3), $w_n(t)\to x(t)$ uniformly. Therefore $x_n(t)\to x(t)$ uniformly.

Now assume $f(t, x)=A(t)x$. Let $X_n(t)$ and $X(t)$ denote fundamental matrices of (4.1) and (4.3), respectively, satisfying $X_n(0)=X(0)=I$. Define $W_n(t)=Y_n^{-1}(t)X_n(t)$. Then $W_n(t)$ satisfies

$$W' = WX_n^{-1}(t)A(t)X_n(t), \qquad W(0) = I. \tag{4.7}$$

Suppose $X_n(t)\to X(t)$ uniformly. From (4.7) $W_n(t)\to W(t)$ uniformly, where $W(t)$ satisfies

$$W' = WX^{-1}(t)A(t)X(t), \qquad W(0) = I. \tag{4.8}$$

By inspection $X(t)$ satisfies (4.8). Hence $W(t)=X(t)$, i.e., $W_n(t)\to X(t)$ uniformly. Therefore $Y_n(t)\to I$ uniformly.

REMARKS. It is not known if (4.5) implies (4.4) in the nonlinear case. The implication "(4.4)⇒(4.5)" seems to be more useful; nonetheless, it would be nice to be able to characterize (4.5) in terms of the simpler (4.4) in the nonlinear case. It would also be nice to characterize (4.4) in terms of $R_n(t)$. However, no such characterization is known. In particular Opial [6] has given an example showing that

$$\left| \int_0^t R_n(s)ds \right| \to 0 \quad \text{uniformly on} \quad [0, T] \tag{4.9}$$

does not imply (4.4). Of course (4.4) and (4.9) are equivalent if $R_n(t)$ is diagonal.

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