STRONGLY BRANCHED COVERINGS OF CLOSED RIEMANN SURFACES[1]

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ABSTRACT. Let b: $W_1 \rightarrow W_2$ be a B-sheeted covering of closed Riemann surfaces of genera p_1 and p_2 respectively. b is said to be *strongly branched* if $p_1 > B^2p_2 + (B-1)^2$. If M_2 is the function field on W_1 obtained by lifting the field from W_2 to W_1, then M_2 is said to be a *strongly branched subfield* if the same condition holds.

If M_1 admits a strongly branched subfield, then there is a unique maximal one. If M_2 is this unique one and f is a function in M_1 so that

$$(B-1)o(f) < (p_1 - Bp_2) + (B-1)$$

then $f \in M_2$, where $o(f)$ is the order of f. (This is a generalization of the hyperelliptic situation.) These results are applied to groups of automorphisms of W_1 to obtain another generalization of the hyperelliptic case.

I. **Introduction.** The main purpose of this paper is to generalize the following properties of hyperelliptic Riemann surfaces. Let W be a hyperelliptic Riemann surface of genus p ($p \geqq 2$) with hyperelliptic conformal self-map T. Let $b: W \rightarrow W/\langle T \rangle$ be the analytic map from W onto the orbit space of $\langle T \rangle$, where $W/\langle T \rangle$ is a Riemann surface of genus zero. Then $\langle T \rangle$ is normal in the full group, $A(W)$, of conformal self-maps of W. Moreover, the quotient group, $A(W)/\langle T \rangle$, is isomorphic to a finite subgroup of $A(W/\langle T \rangle)$. Finally, if f is a meromorphic function on W of order no greater than p, then f is the lift, via the projection b, of a rational function on $W/\langle T \rangle$.

Let $b: W_1 \rightarrow W_2$ be an analytic map of closed Riemann surfaces of B sheets and total ramification r_b. If p_i is the genus of W_i for $i=1, 2$ then the Riemann-Hurwitz formula reads

$$2p_1 - 2 = B(2p_2 - 2) + r_b. \tag{1}$$

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The map b will be called *strongly branched* if

$$r_b > 2B(B-1)(p_2+1). \tag{2}$$

Notice that for $B=2$, $p_2=0$, and $r_b>4$ we are in the hyperelliptic case, for then $p_1>1$. The aspect of the hyperelliptic situation we are stressing then is that for fixed p_2 and B, r_b is large. Note that the identity map is not strongly branched.

A conformal self-map of a Riemann surface will be called an *automorphism*. The full group of automorphisms of a surface W will be denoted $A(W)$. Our hypotheses will always imply that the genus of W is at least two and so $A(W)$ will be a finite group. If G is a subgroup of $A(W)$ then the space of orbits of G, W/G, is naturally a Riemann surface so that the projection $W\to W/G$ is an analytic map and the number of sheets is equal to the order of G. One of the generalizations of the hyperelliptic case, Corollary 3, is as follows. If G is a simple group and the map $W\to W/G$ is strongly branched then G is normal in $A(W)$.

Again let $b: W_1\to W_2$ be an analytic map which is not necessarily strongly branched. Continue the notation of the previous paragraph. Let f_1 be a meromorphic function on W_1 of order $o(f_1)$. Let $f_1^1, f_1^2, \cdots, f_1^B$ be the branches of f_1 on the B sheets of the covering. Define $D_b(f_1)$ as follows.

$$D_b(f_1) = \prod_{i<j} (f_1^i - f_1^j)^2.[2] \tag{3}$$

Then $D_b(f_1)$ is a well-defined function on W_2 whose order is at most $2(B-1)o(f_1)$. The number of zeros of $D_b(f_1)$ is at least r_b.[3] Thus if $r_b>2(B-1)o(f_1)$ then $D_b(f_1)\equiv 0$. The following lemma follows from well-known principles.

LEMMA 1.[4] *If $r_b>2(B-1)o(f_1)$ then the map $b: W_1\to W_2$ admits a factoring $c: W_1\to W_3$ and $d: W_3\to W_2$ where $b=d\circ c$, c is not one-to-one, and there is a function f_3 on W_3 so that $f_1=f_3\circ c$.*

The definition of strongly branched is designed to assure that the hypothesis of Lemma 1 holds for functions on W_1 whose order is at most $B(p_2+1)$. Notice that if B is a prime integer then the map d of

[2] $D_b(f_1)$ is usually called the discriminant. See Hensel-Landsberg [**3**, p. 26].

[3] If the poles of f_1 lie on branch points of b this need not be correct. Rather than make a more complicated computation it is better to replace f_1 by Tf_1 where T is a fractional linear transformation. The results will then hold for Tf_1. But the nature of all results of this paper is that they will then hold for f_1.

[4] The author first observed this technique in Röhrl [**4**].

Lemma 1 must be one-to-one. For composite B, however, a nontrivial factorization may occur. In this latter case it would appear that the intermediate surface, W_3, depends on the choice of f_1. The main result of this paper, Lemma 4, states that a W_3 can be chosen independently of any f_3 satisfying the hypotheses of Lemma 1. Moreover, W_3 can be chosen uniquely.

The methods of this paper give fairly simple proofs of some well-known results. We include proofs of some classical results on linear series, the usual proofs being somewhat inaccessible. We also give constructive examples of closed surfaces of genus five or more which admit only the identity automorphism.[5]

The results of this paper can be expressed in terms of function fields, and we will find it convenient to do so. If $b: W_1 \rightarrow W_2$ is a B-sheeted covering, let M_1 be the field of meromorphic functions on W_1 and let M_2 be that field in M_1 obtained by lifting all functions from W_2 to W_1 via b. The index of M_2 in M_1 is B. Then there is a one-to-one correspondence between nonconstant subfields of M_1 and surfaces of the type W_2. Moreover, if fields are ordered by inclusion and surfaces are ordered by the obvious covering requirement then this correspondence preserves these partial orderings. Consequently, we will call a subfield M_2 a *strongly branched subfield* if the covering $W_1 \rightarrow W_2$ is strongly branched.

Using the Riemann-Hurwitz formula (1) and the definition of strongly branched (2) we obtain other criteria for strongly branched which are listed in the next lemma. Criterion (i) allows one to define the concept without reference to the ramification. Criterion (ii) is useful in the proofs.

LEMMA 2. *If $b: W_1 \rightarrow W_2$ is a covering of closed surfaces then the following conditions are equivalent to b being strongly branched*:

(i) $p_1 > B^2 p_2 + (B-1)^2$,[6]

(ii) $Br_b > (B-1)(2p_1 - 2 + 4B)$.

II. Some classical results.

We now use the ideas introduced in §I to prove the classical results mentioned earlier.

PROPOSITION I. *Let W_1 be a Riemann surface of genus p_1. Let f and g be two meromorphic functions on W_1 of orders F and G respectively. If f, g generate the full field of functions on W_1 then*

[5] The proof that there are surfaces of every genus ($p \geqq 3$) with trivial automorphism groups will be found in Baily [1].

[6] The consequences of criterion (i) with $p_2 = 0$ are discussed by Dorodnov [2]. For B prime part of the results of this paper are included in his.

$$p_1 \leqq (F-1)(G-1).^7$$

PROOF. Consider $f: W_1 \to W_2$ where W_2 is the Riemann sphere and r_f is the total ramification. Since the field generated by f and g separates points of W_1, it follows that $D_f(g)$ cannot be identically zero.[8] Thus $r_f \leqq 2(F-1)G$. But $2p_1-2=-2F+r_f$. The result follows immediately. q.e.d.

PROPOSITION II. *Let W_1 be a Riemann surface of genus p_1. Let the constant function* 1, *f, and g be three linearly independent meromorphic functions on W_1 so that f and g have the same order F. Suppose further that the poles of f and g are simple and they coincide. If f and g generate the field of functions on W_1 then*

$$p_1 \leqq \tfrac{1}{2}(F-1)(F-2).^9$$

PROOF. The proof is the same as that of Proposition I except that the added information about the poles of f and g allows one to deduce that $o(D_f(g)) \leqq F(F-1)$, and so $r_f \leqq F(F-1)$. q.e.d.

Propositions I and II can be interpreted to give conditions when two functions do not generate the function field on W_1.

COROLLARY 1. *Assume all hypotheses of Proposition* I *except the last. Assume that $p_1 > (F-1)(G-1)$. Then f and g generate a strongly branched subfield.*

COROLLARY 2. *Assume all hypotheses of Proposition* II *except the last. Assume that $p_1 > \frac{1}{2}(F-1)(F-2)$. Then f and g generate a strongly branched subfield.*

PROOF. Since the proofs of the corollaries are almost identical we include only a proof of Corollary 2. Let M_2 be the field generated by f, g and the constants and let M_1 be the full field on W_1. Let the B-sheeted covering $b: W_1 \to W_2$ correspond to the inclusion of fields

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[7] See Hensel-Landsberg [3, p. 385]. The method of proof presented here is essentially that of Hensel-Landsberg. It will be understood that the field generated by f and g will include the constant field, $\boldsymbol{C}$.

[8] We may again assume that no pole of g lies on a branch point of f.

[9] The hypotheses of Proposition II make most sense in the context of linear series. Let D be the polar divisor of f. If we consider those divisors equivalent to D which are parametrized by the linear family generated by f, g and the constant function 1, we obtain a linear series of dimension 2 and order F, denoted g_F^2. To say that "f, g and 1 generate M_1" is rendered in the usual terminology by "g_F^2 is simple." Here the triple $(f, g, 1)$ gives a birational map of W_1 onto a plane curve of order F. Proposition II is simply the fact that the genus of a plane algebraic curve of order F is bounded by $\frac{1}{2}(F-1)(F-2)$. See Hensel-Landsberg [3, p. 427] or Walker [5, p. 179].

$M_2 \subset M_1$. Let p_2 be the genus of W_2. f and g arise from functions on W_2 of order F/B. By Proposition II

$$p_2 \leqq \tfrac{1}{2}(F/B - 1)(F/B - 2),$$

so

$$B^2 p_2 + (B - 1)^2 \leqq \tfrac{1}{2}(F - B)(F - 2B) + (B - 1)^2.$$

The corollary will follow by criterion (i) of Lemma 2 if

$$\tfrac{1}{2}(F - B)(F - 2B) + (B - 1)^2 \leqq \tfrac{1}{2}(F - 1)(F - 2).$$

That is $0 \leqq 3BF - 3F - 4B^2 + 4B$ or $0 \leqq (B-1)(3F - 4B)$.

Since $1, f$, and g are linearly independent,

$$2 \leqq B \leqq F/2.$$

The result now follows. q.e.d.

That strongly branched coverings arise in this context perhaps indicates that the concept is a little less artificial than the original ad hoc definition might indicate.

III. **Main results.** We shall now adopt the following notational convention. Analytic maps will be denoted by small letters: $b, c, \cdots$; the number of sheets denoted by the corresponding capital letters: $B, C, \cdots$; and total ramification denoted: $r_b, r_c, \cdots$. Surfaces W_i will have genus p_i for $i = 1, 2, \cdots$. All surfaces will be covered by W_1. The function field on W_1 will be denoted by M_1. If $b: W_1 \to W_i$ then M_i $(\subset M_1)$ will be the function field of index B obtained by lifting functions from W_i to W_1 via b, $i = 2, 3, \cdots$.

LEMMA 3. *Let $b: W_1 \to W_2$, $c: W_1 \to W_3$, $d: W_3 \to W_2$ be analytic maps such that $b = d \circ c$. If b is strongly branched then either c or d is.*

PROOF. Assume that neither c nor d is strongly branched. Then $p_1 \leqq C^2 p_3 + (C-1)^2$ and $p_3 \leqq D^2 p_2 + (D-1)^2$. Therefore

$$p_1 \leqq (CD)^2 p_2 + C^2(D - 1)^2 + (C - 1)^2.$$

Since $B = CD$ it suffices to show that

$$C^2(D - 1)^2 + (C - 1)^2 \leqq (CD - 1)^2$$

in order that b be not strongly branched. This is straightforward. q.e.d.

DEFINITION. A strongly branched subfield M_2 of M_1 will be called a *maximal strongly branched subfield* of M_1 if whenever $M_2 \subset M_3 \subset M_1$, $M_2 \neq M_3$ then M_3 is not a strongly branched subfield. The corresponding definition will also hold for coverings.

LEMMA 4. *Let $b: W_1 \to W_2$ be a maximal strongly branched covering. Suppose f_1 is a funciion on W_1 so that $2(B-1)o(f_1) < r_b$.*
Then there is an f_2 on W_2 so that $f_1 = f_2 \circ b$ (i.e., $f_1 \in M_2$).

PROOF.[10] Lemma 1 assures us of a factorization $c: W_1 \to W_3$, $d: W_3 \to W_2$ where $b = d \circ c$, c is not one-to-one, and there is an f_3 on W_3 so that $f_1 = f_3 \circ c$. Choose W_3 so that D is minimum. If $D > 1$ we will show that $2(D-1)o(f_3) < r_d$. Another application of Lemma 1 will give a factorization of d and a contradiction of the minimality of D. Then the proof will be complete.

So assume $D > 1$. Now b is strongly branched but c is not. Consequently d is strongly branched. This means

$$r_c \leqq 2C(C-1)(p_3+1) \tag{4}$$

and

$$r_d > \frac{D-1}{D}(2p_3 - 2 + 4D). \tag{5}$$

Thus $r_c \leqq C(C-1)(2p_3 - 2 + 4D) - 4C(C-1)(D-1)$ or

$$r_c < \frac{CD(C-1)}{(D-1)} r_d. \tag{6}$$

Several applications of the Riemann-Hurwitz formula give $r_b = r_c + Cr_d$ which together with (6) and the fact that $B = CD$ yields $r_b < (C + B(C-1)/(D-1))r_d$ or

$$(D-1)r_b < C(B-1)r_d. \tag{7}$$

Since $Co(f_3) = o(f_1)$ we have

$$2(D-1)o(f_3) = \frac{2(D-1)}{C} o(f_1) < \frac{(D-1)}{C(B-1)} r_b < r_d. \qquad \text{q.e.d.}$$

LEMMA 5. *If a maximal strongly branched subfield of M_1 exists, then it is unique.*

PROOF. Suppose $b: W_1 \to W_2$ and $c: W_1 \to W_3$ are two maximal strongly branched coverings. We may assume that $B(p_2+1) \geqq C(p_3+1)$. Let f_3 be a function on W_3 of order no greater than p_3+1. Then $f_3 \circ c$ is in M_3 and has order no greater than $C(p_3+1)$. Thus, by Lemma 4, $f_3 \circ c \in M_2$. Since M_3 is generated by functions of order

[10] The author is indebted to W. T. Kiley for showing how the original version of this paper could be considerably simplified. In particular, the idea of the proof of Lemma 4 is due to him.

$C(p_3+1)$ or less, we have $M_3 \subset M_2 \subset M_1$. Since M_3 is maximal it follows that $M_3 = M_2$. q.e.d.

We will summarize the above results in the following theorem.

THEOREM 1. *Let M_1 be the field of functions on a Riemann surface of genus p_1. Suppose M_1 admits a strongly branched subfield. Then there exists a unique maximal strongly branched subfield M_2 of genus p_2 and index B. All strongly branched subfields lie in M_2. If f_1 is a function in M_1 so that either*

(i) $(B-1)o(f_1) < (p_1 - Bp_2) + (B-1)$ *or*

(ii) $o(f_1) \leqq B(p_2+1)$,

then $f_1 \in M_2$.

PROOF. If M_1 admits a strongly branched subfield M_3 then there is a maximal strongly branched subfield M_2 so that $M_3 \subset M_2 \subset M_1$. Since M_2 is unique the first two assertions of the theorem are proved. Note that (ii) implies (i) and that (i) is just Lemma 4 with r_a replaced using the Riemann-Hurwitz formula. q.e.d.

IV. **Applications to automorphisms of Riemann surfaces.** Let W_1 be a Riemann surface whose function field, M_1, admits a maximal strongly branched subfield M_2. Let $W_1 \rightarrow W_2$ be the corresponding covering. Any automorphism of W_1, acting on M_1, leaves M_2 invariant, since M_2 is unique. Let N be the subgroup of $A(W_1)$ which leaves the functions of M_2 pointwise fixed. Then N is easily seen to be a normal subgroup of $A(W_1)$ and is, in fact, the group of cover transformations[11] of the cover $W_1 \rightarrow W_2$. The quotient group $A(W_1)/N$ is isomorphic to a subgroup of $A(W_2)$. Moreover, this subgroup of $A(W_2)$ must respect the branching of the covering $W_1 \rightarrow W_2$ in the obvious manner. Finally, if M_1 is a Galois extension of M_2 then N is the corresponding Galois group and N has order equal to the index of M_2 in M_1. We summarize this discussion in the following theorem which can be viewed as a generalization of the hyperelliptic case.

THEOREM 2. *Suppose W_1 admits a group of automorphisms whose fixed field is a strongly branched subfield. Then M_1 is a Galois extension of the maximal strongly branched subfield whose Galois group is normal in $A(W_1)$.*

COROLLARY 3. *If W_1 admits a simple group of automorphisms whose fixed field is strongly branched, then this group is normal in $A(W_1)$.*

[11] In the context of a branched covering b: $W_1 \rightarrow W_2$ a cover transformation T is an automorphism of W_1 such that $b \circ T = b$.

PROOFS. If M_3 is a strongly branched subfield and M_1 is a Galois extension of M_3 then M_1 is a Galois extension of the maximal strongly branched subfield, M_2, since $M_3 \subset M_2 \subset M_1$. If the Galois group of M_1 over M_3 is simple then $M_2 = M_3$ since otherwise the simple group would contain a nontrivial normal subgroup. q.e.d.

The above discussion can also be used to construct Riemann surfaces which admit only the identity as an automorphism. Let W_2 be a Riemann surface and let T be a finite set on W_2 so that the only automorphism of W_2 which permutes T is the identity. Now let $b: W_1 \to W_2$ be a strongly branched covering branched only over T and such that the only cover transformation is the identity. Then in the discussion preceding Theorem 2, N and $A(W_1)/N$ must both be the identity, and so also then is $A(W_1)$. To be more specific, let $p_2 = 0$, $B = 3$ and $r_b > 12$. Then the covering b will be strongly branched. To insure that $N = \{\text{id}\}$ it suffices to have at least one branch point of multiplicity 2 since the only candidate for the Galois group is Z_3. Choose the points T so that no fractional linear transformation permutes them. Since the order of T is more than 4 this can be achieved. The surfaces thus constructed will have genus $p_1 > 4$.

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