SOME EXAMPLES OF RIGHT SELF-INJECTIVE RINGS WHICH ARE NOT LEFT SELF-INJECTIVE

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ABSTRACT. Endomorphism rings of nonfinitely generated free right R-modules over a quasi-Frobenius ring R are right self-injective but not left self-injective.

A well-known example of a right self-injective ring which is not left self-injective is the full ring of linear transformations of an infinite dimensional right vector space over a division ring, with linear transformations acting on the left of vectors, see e.g. [4].

The purpose of this note is to show that the class of rings which are right self-injective but not left self-injective includes the endomorphism rings of nonfinitely generated free right modules over a quasi-Frobenius ring.

All rings have unity element and all modules are unital.

THEOREM 1. *Let M_R be a right R-module, R a ring, which is an infinite direct sum of nonzero submodule $\{M_i \mid i \in I\}$, I some index set. For $S = \mathrm{Hom}(M_R, M_R)$ such that ${}_SM_R$, ${}_SM$ is not an injective S-module.*

PROOF. Let $e_i: M_R \to M_i$ be the ith projection of the module M_R onto the submodule M_i, then $\{e_i \mid i \in I\}$ is an infinite set of orthogonal idempotents of S. Let ${}_SA$ be the left ideal of S generated by $\{e_i \mid i \in I\}$. Since $M_i \neq 0$ for each $i \in I$, choose $0 \neq x_i \in M_i$. Clearly there is an S-homomorphism $f: {}_SA \to {}_SM$ such that $e_i f = x_i = e_i x_i$. If f were extendable to a homomorphism from ${}_SS$ to ${}_SM$, then it would be given by a right multiplication by some element of M. However for any element $x \in M$, $e_i x = 0$ for all but finitely many $i \in I$, so f is not extendable to ${}_SS$, so ${}_SM$ is not injective.

THEOREM 2. *Let R be a quasi-Frobenius ring and F_R a nonfinitely generated free right R-module and $S = \mathrm{Hom}(F_R, F_R)$, so that ${}_SF_R$, then S is right self-injective but not left self-injective.*

PROOF. By [3, Theorem A], F_R is injective since it is projective, hence for A_S, ${}_SF_R$, F_R

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$$\begin{aligned} \mathrm{Ext}^1_S(A_S, S_S) &= \mathrm{Ext}^1_S(A_S, \mathrm{Hom}(F_R, F_R)_S) \\ &\cong \mathrm{Hom}(\mathrm{Tor}^S_1(A_S, {}_SF)_R, F_R) \end{aligned} \tag{1}$$

by [2, Proposition VI. 5.1]. Since F_R is a generator ${}_SF$ is finitely generated projective by [1, Theorem 2], hence $\mathrm{Tor}^S_1(A_S, {}_SF) = 0$, so by (1), $\mathrm{Ext}^1_S(A_S, S_S) = 0$ for each A_S, from which it follows that S is right self-injective by [2, Proposition VI. 2.1a].

By Theorem 1, ${}_SF$ is not injective hence S is not left self-injective since ${}_SF$ is finitely generated projective (i.e. a direct summand of a finite number of copies of ${}_SS$).

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