STABLE HOMOTOPY THEORY IS NOT SELF-DUAL[1]

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ABSTRACT. The classical Spanier-Whitehead duality for finite complexes shows that the finite stable homotopy category is self-dual. We prove that in the larger stable categories, duality is not consistent with the standard arguments of homotopy theory.

One motivation for introducing stable homotopy theory is Spanier-Whitehead duality [2]. This duality implies that a certain stable category formed from finite cell complexes is self-dual. For other purposes, however, this stable category has to be enlarged in some way. We prove that Spanier-Whitehead duality does not hold in the larger category of spectra $\mathcal{S}_h$ introduced in [1]. Our proof shows more, that this failure is inherent in any stable theory rich enough to allow the usual procedures of homotopy theory.

We refer to [1] freely. Finite spectra X enjoy the property [1, C26] that $\{X, \bigvee_i Y_i\} = \bigoplus_i \{X, Y_i\}$ for any family of spectra Y_i, in other words (see [1, C20]) that any morphism from X to an infinite sum factors through some finite subsum. Let us call a spectrum X with the dual property "cofinite", that is, $\{\prod_i Y_i, X\} = \bigoplus_i \{Y_i, X\}$ for any family of spectra Y_i (products exist by [1, C22]). We find there are not enough cofinite spectra for duality.

THEOREM. *The category $\mathcal{S}_h$ is not equivalent to its dual category. In fact the only cofinite spectra are contractible.*

PROOF. Suppose X is a noncontractible cofinite spectrum.

(1) By [1, L5], X has a nonzero homotopy group.

(2) We take $Y_i = S^i$, the i-sphere, for $i > 0$. Then the wedge $\bigvee_i S^i$ also serves as the product $\prod_i S^i$ in $\mathcal{S}_h$ by [1, L7], so that $\{\prod_i S^i, X\} = \prod_i \{S^i, X\}$. It follows that $\pi_i(X) = 0$ for all sufficiently large i.

(3) Let $G = \pi_n(X)$ be the top nonzero homotopy group of X, which exists by (1) and (2). By obstruction theory, $\{Y, X\} \cong \operatorname{Hom}(\pi_n(Y), G)$ whenever Y is $(n-1)$-connected. Since $\pi_n(Y)$ is arbitrary and we can choose each Y_i to be $(n-1)$-connected, we find

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$$\operatorname{Hom}\left(\prod_i H_i, G\right) = \bigoplus_i \operatorname{Hom}(H_i, G)$$

for any family of abelian groups H_i. This imposes strong conditions on G.

(4) G does not contain the cyclic group $\boldsymbol{Z}_p$ as a subgroup, for any prime p. Take $H_i = \boldsymbol{Z}_p$ for all i, so that $\bigoplus_i H_i \subset \prod_i H_i$ are vector spaces over the field $\boldsymbol{Z}_p$. Then any homomorphism $\bigoplus_i H_i \to G$ extends over $\prod_i H_i$, which contradicts (3) unless $\operatorname{Hom}(\boldsymbol{Z}_p, G) = 0$. Hence G is torsion-free.

(5) G does not contain the group $\boldsymbol{Q}$ of rationals, by replacing $\boldsymbol{Z}_p$ by $\boldsymbol{Q}$ throughout (4).

(6) If $A \to B \to C \to A$ is an exact triangle and A and B are cofinite, then by the five-lemma C is also cofinite. We take $A = B = X$ and $\gamma: A \to B$ as p times the identity class, which we may do by [1, J11]. (We construct C as the mapping cone of γ.) Then by exactness, since $\gamma_*: \pi_*(X) \to \pi_*(X)$ is also multiplication by p, $\pi_*(C)$ consists only of p-torsion, which contradicts (4) for C unless $\pi_*(C) = 0$. Thus γ_* must be an isomorphism, and the group G must be divisible.

From (4), (5) and (6) we have a contradiction on G, since any torsion-free divisible group is a rational vector space. Thus G and X do not exist.

References

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