OSCILLATION THEOREMS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Recently, Bobisud, Proc. Amer. Math. Soc. **23** (1969), 501–505, has discussed the oscillatory behavior of solutions of $y''+a(t)f(y)=0$. The purpose of this paper is to prove Bobisud's theorems under weaker assumptions. The proofs are quite different from his and simpler.

1. **Introduction.** Recently, Bobisud [2] has discussed the oscillatory behavior of solutions of

$$y'' + a(t)f(y) = 0. \tag{1}$$

Let S denote the set of all solutions of (1) which exist on some positive half-line $[T_y, \infty)$, where T_y depends on the particular solution y. A solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros, and (1) is said to be *oscillatory* if every solution of (1) is oscillatory.

With $f(y)\equiv y^{2n+1}$ $(n\geqq 1)$ and $a>0$, Atkinson [1] proved that all solutions of (1) are oscillatory if and only if

$$\int_x^\infty ta(t)dt = +\infty. \tag{2}$$

Many authors (cf. [3], [4]) have extended the results of Atkinson, and Bobisud [2] has proved oscillation theorems for (1) without the restriction $a>0$.

The purpose of this paper is to show that Theorem 1 and Theorem 2 [2] remain valid without the assumption

$$0 < \lim_{t\to+\infty} \int_x^t a(s)ds \leqq +\infty \tag{A}$$

and to prove Theorem 3 [2] under weaker conditions. The proofs of our theorems are quite different from Bobisud's and moreover very easy.

2. **Oscillation theorems.**

THEOREM 1. *Let f be continuous and continuously differentiable on $(-\infty, 0)\cup(0, \infty)$ with $uf(u)>0, f'\geqq 0$ there; let f also satisfy*

Received by the editors March 10, 1970.

AMS 1969 *subject classifications.* Primary 3442; Secondary 3450.

Key words and phrases. Oscillatory, monotonically to zero.

(B) $\int_{x}^{\infty} du/f(u) < \infty, \quad \int_{-x}^{-\infty} du/f(u) < \infty, \qquad$ *for every* $x > 0$.

Finally, assume

$$\int_{x}^{\infty} sa(s)ds = +\infty.$$

Then every $y \in S$ *is either oscillatory or tends monotonically to zero, as* $t \to \infty$.

PROOF. Let $y \in S$ be nonoscillatory and assume that $y > 0$, for some $p > 0$ on $[p, \infty)$; a similar argument treats the case $y < 0$. Thus, from (1), we obtain

$$ty''/f(y) = -ta(t).$$

By integration from p to t and integration by parts, we obtain

$$\frac{ty'(t)}{f(y(t))} = \int_{p}^{t} \frac{y'(t)}{f(y(t))} dt - \int_{p}^{t} \frac{tf'(y(t))(y'(t))^2}{(f(y(t)))^2} dt + \frac{py'(p)}{f(y(p))} - \int_{p}^{t} ta(t)dt.$$

Since $f'(y(t)) \geqq 0$, we find

$$\frac{ty'(t)}{f(y(t))} \leqq \int_{y(p)}^{y(t)} \frac{1}{f(s)} ds - \int_{p}^{t} sa(s)ds + \frac{py'(p)}{f(y(p))}. \tag{3}$$

The first integral in the right-hand side is bounded above, by the hypothesis (B). Thus, (3) implies that

$$\frac{ty'(t)}{f(y(t))} \to -\infty, \quad \text{as } t \to \infty.$$

So, we obtain for some $k > 0$,

$$\frac{ty'(t)}{f(y(t))} < -k \qquad \text{for every } t \geqq \exists\, q \geqq p. \tag{4}$$

By integration from q to t, we obtain for $t \geqq q$,

$$\int_{y(q)}^{y(t)} \frac{1}{f(s)} ds < \ln\left(\frac{q}{t}\right)^k. \tag{5}$$

From (5), we conclude $\lim_{t\to\infty} y(t) = 0$. Thus, we obtain that $y \in S$ is oscillatory or tends monotonically to zero as $t \to \infty$.

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *assume that for some* $x>0$,

$$\text{(C)} \qquad \lim_{t\to 0+}\int_t^x \frac{du}{f(u)} < \infty, \qquad \lim_{t\to 0-}\int_t^{-x} \frac{du}{f(u)} < \infty.$$

Then every $y\in S$ *is oscillatory.*

PROOF. Proceeding as above, we have only to show that $y\in S$ does not tend monotonically to zero, as $t\to\infty$. This follows immediately from (5), since the left-hand side of (5) is bounded below by (C).

THEOREM 3. *In addition to the hypotheses of Theorem* 1, *assume that* $f'\geqq\gamma>0$ *on* $(-\infty, 0)\cup(0, \infty)$, *that* $a(s)$ *and* $a_-(s)$ $(\equiv\min(a(s), 0))$ *is integrable over* $(0, \infty)$, *and that*

$$0\leqq \lim_{t\to\infty}\int_x^t a(s)ds + \frac{\gamma^3}{x} \leqq +\infty.$$

Then every $y\in S$ *is oscillatory.*

PROOF. As in the proof of Theorem 2, we have only to show that $y\in S$ does not tend monotonically to zero, as $t\to\infty$. Assume then that $y>0$, $y'<0$ on $[p, \infty)$; a similar argument treats the case $y<0$, $y'>0$. From (4), we obtain, for some $\gamma>0$,

$$\text{(6)} \qquad \frac{y'(t)}{f(y(t))} < -\frac{\gamma}{t}, \qquad \text{for all } t\ (\geqq q \ni\ \geqq p).$$

Thus,

$$\text{(7)} \qquad \int_q^t f'(y(s))\left[\frac{y'(s)}{f(y(s))}\right]^2 ds > \int_q^t \frac{\gamma^3}{s^2}\,ds = \gamma^3\left[\frac{1}{q}-\frac{1}{t}\right].$$

From (1), it follows (cf. [2, p. 502]) that

$$\text{(8)} \quad \frac{y'(t)}{f(y(t))} - \frac{y'(q)}{f(y(q))} + \int_q^t f'(y(s))\left[\frac{y'(s)}{f(y(s))}\right]^2 ds + \int_q^t a(s)ds = 0.$$

By means of Theorem 3 and (7), we obtain from (8)

$$\lim_{t\to\infty}\frac{y'(t)}{f(y(t))} < 0.$$

This leads to a contradiction as, in Bobisud's argument [2, p. 504], and the proof is complete.

THEOREM 4. *Let f be continuous and continuously differentiable on* $(-\infty, 0)\cup(0, \infty)$ *with* $uf(u)>0, f'\geqq 0$ *there; let f also satisfy*

$$\text{(D)} \qquad \int_0^{\infty} du/f(u) < \infty , \qquad \int_0^{-\infty} du/f(u) < \infty .$$

Assume that $a(t)$ is continuous and $ta-(t)$ is integrable. Then $y\in S$ is oscillatory if and only if

$$\int_x^{\infty} ta(t)dt = +\infty .$$

PROOF. The necessity is included in Theorem 2. The sufficiency follows immediately from the proof of Theorem 1 [4, p. 113].

THEOREM 5. *In addition to the hypotheses of Theorem* 1, *assume that $a(t)$ be eventually nonnegative. Then every $y\in S$ is oscillatory.*

PROOF. Let $y\in S$ be nonoscillatory. Then we may without loss of generality assume that $y(t)>0$ for $t\geqq {}^{\exists}q\geqq p$. From (1), we obtain $y''\leqq 0$. Hence $y'(t)\geqq 0$ for $t\geqq q$, which contradicts (4).

REMARKS. Macki and Wong [4] have discussed Theorem 4 in the case $a(t)>0$. Theorem 5 shows that Theorem 2 [3] is valid for $n=1$. Although Theorem 5 is included in Macki-Wong's Theorem [4], it is important that Theorem 5 is extended to arbitrary even order differential equations (cf. [3]).

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