PEAK POINTS FOR HYPO-DIRICHLET ALGEBRAS

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ABSTRACT. A proof is given of a theorem limiting the number of points not in the Choquet boundary for certain uniform algebras. In the hypo-Dirichlet case the argument reduces to a short proof that every point is in the Choquet boundary. Also, a lemma is presented which describes, for certain subalgebras of codimension one in a uniform algebra, a relation between certain spaces of real functions associated with the original algebra, and the corresponding spaces associated with the subalgebra.

Ahern and Sarason have shown that if A is a hypo-Dirichlet algebra on X then X is the Choquet boundary of A [**AS**, Corollary 2 to Theorem 3.1]. They obtain this as a consequence of a version of the F. and M. Riesz theorem, and comment that they know of no essentially simpler proof. In this note I prove a slightly more general theorem, whose proof reduces in the hypo-Dirichlet case (when the closure of Re A has finite codimension in $C_R(X)=S_A$, in the terminology below) to a short and direct proof that X is the Choquet boundary of A. Our proof of the theorem is in the form of two simple lemmas, and Lemma 1 is not needed in the hypo-Dirichlet case.

Throughout, X will denote a compact Hausdorff space. If $x\in X$, δ_x denotes point mass at x and γ_x denotes the functional $\gamma_x(f)=f(x)$ on $C(X)$. If L is a point-separating subspace of $C(X)$ or $C_R(X)$ containing the constants, for each $x\in X$ we denote by $\mathfrak{M}_x(L)$ the set of positive regular Borel measures μ on X such that $\int f d\mu=f(x)$ for all $f\in L$; always $\delta_x\in\mathfrak{M}_x(L)$, and if $\mathfrak{M}_x(L)=\{\delta_x\}$ we say that $x\in \mathrm{Ch}(L)$ the Choquet boundary of L. Clearly L and its closure have the same Choquet boundary.

LEMMA 1. *Let L be a subspace of $C_R(X)$ containing the constants and separating the points of X. Suppose*

(1) *If $x_0\in X$, $\gamma_{x_0}|L$ is not a finite convex combination of the functionals $\gamma_x|L$ for $x_0\neq x\in X$.*

(2) *L has finite codimension k in $C_R(X)$.*

Then there are at most k points in $X\backslash \mathrm{Ch}(L)$.

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PROOF. We first observe that if s is a positive integer and G_s denotes the (real or complex)-entried $s\times s$ matrices $T=(t_{ij})$ endowed with its usual structure as a (real or complex) algebra, the function

$$\|T\| = \max_{1\leq j\leq s}\left(\sum_{i=1}^{s}|t_{ij}|\right)$$

is a Banach algebra norm on G_s inducing the usual topology on G_s.

Suppose now that $k<s<\infty$ and $x_1, \cdots, x_s$ are distinct points in $X\backslash\mathrm{Ch}(L)$. For $1\leqq j\leqq s$ let $\delta_{x_j}\neq\mu_j\in\mathfrak{M}_{x_j}(L)$ and write $\mu_j=\sum_{i=1}^{s} t_{ij}\delta_{x_i}+\nu_j$ where $t_{ij}\geqq 0$ and ν_j has no mass at $x_1, \cdots, x_s$. Let $T=(t_{ij})\in G_s$, so $\|T\|\leqq 1$.

Each $\delta_{x_j}-\mu_j$ is a real measure orthogonal to L. Since $s>k$, there are real numbers $c_1, \cdots, c_s$ not all 0 such that $\sum_{j=1}^{s} c_j(\delta_{x_j}-\mu_j)=0$.

Looking at the coefficient of δ_{x_i} in this sum we obtain $c_i-\sum_{j=1}^{s} t_{ij}c_j=0$ for all i or, letting Γ denote the column matrix (c_i), $\Gamma=T\Gamma$. Iteration gives $\Gamma=T^r\Gamma$ for every positive integer r, and since $\Gamma\neq 0$ we conclude that $\|T\|\geqq 1$. Thus $\|T\|=1$ and there is J with $\sum_{i=1}^{s} t_{iJ}=1$, hence $\nu_J=0$. Since $\mu_J\neq\delta_{x_J}$, $t_{JJ}<1$ and

$$(1-t_{JJ})^{-1}(\mu_J-t_{JJ}\delta_{xJ}) = \sum_{i=1;i\neq J}^{s} t_{iJ}(1-t_{JJ})^{-1}\delta_{x_i}.$$

Integrating against functions in L, this becomes

$$\gamma_{x_J}\Big| L = \sum_{i=1;i\neq J}^{s} t_{iJ}(1-t_{JJ})^{-1}\gamma_{x_i}\Big| L,$$

contradicting (1). $\square$

REMARK 1. Obviously the lemma admits generalization; if L is any subspace of $C_R(X)$ whose closure has finite codimension, then only finitely many points of X have positive measures other than point mass representing them.

REMARK 2. Let D be a relatively compact open set in an open Riemann surface S, $\overline{D}$ its closure and ∂D its boundary. Let $L^0(D)$ be the functions in $C_R(\overline{D})$ harmonic on D, and $L(D)=L^0(D)|\partial D$. $\mathrm{Ch}(L(D))$ is easily seen to consist entirely of regular points for the Dirichlet problem on D. Given $x\in S$ there is a holomorphic function on S whose only zero is x; from this fact it is immediate that (1) is satisfied by $L(D)$. Thus, if $L(D)$ has finite codimension k in $C_R(\partial D)$, then all but at most k points of ∂D are regular for the Dirichlet problem on D.

Now suppose A is a uniform algebra on X, i.e., a uniformly closed point-separating subalgebra of $C(X)$ containing the constants. Let L_A denote the largest (automatically closed) subspace of $C_R(X)$

contained in the closure of $\log|A^{-1}|$, and let S_A denote the closed span of $\log|A^{-1}|$ in $C_R(X)$. Evidently $\operatorname{Re} A \subset L_A \subset S_A$.

THEOREM. *Let A be a uniform algebra on X and suppose L_A has finite codimension in $C_R(X)$. Then the number of points in $X \backslash \mathrm{Ch}(A)$ does not exceed the codimension of S_A in $C_R(X)$.*

The proof is obtained by combining Lemmas 1 and 2. We preface Lemma 2 with two additional remarks which hold for any uniform algebra A on X.

REMARK 3. Since A interpolates all complex-valued functions on finite subsets of X, the set $\{\gamma_x|\operatorname{Re} A : x \in X\}$ and *a fortiori* the sets $\{\gamma_x|L_A : x \in X\}$ and $\{\gamma_x|S_A : x \in X\}$ are linearly independent, so $\operatorname{Re} A$, L_A and S_A satisfy (1).

REMARK 4. Evidently $\mathfrak{M}_x(S_A) \subset \mathfrak{M}_x(L_A) \subset \mathfrak{M}_x(\operatorname{Re} A) = \mathfrak{M}_x(A)$ for $x \in X$, and by Hoffman's uniqueness argument [**B**, 4.1.1] $\mathfrak{M}_x(A) \subset \mathfrak{M}_x(L_A)$, so

$$(3) \qquad \mathfrak{M}_x(S_A) \subset \mathfrak{M}_x(L_A) = \mathfrak{M}_x(\operatorname{Re} A) = \mathfrak{M}_x(A), \qquad \forall x \in X.$$

From this we get immediately

$$(4) \qquad \mathrm{Ch}(A) = \mathrm{Ch}(\operatorname{Re} A) = \mathrm{Ch}(L_A) \subset \mathrm{Ch}(S_A).$$

Alternatively, we may obtain $\mathrm{Ch}(L_A) \subset \mathrm{Ch}(A)$ by invoking [**B**, 2.2.6 and 2.3.4] as in the proof of the lemma below.

LEMMA 2. *Let A be a uniform algebra on X. Suppose L_A has finite codimension in S_A. Then* $\mathrm{Ch}(A) = \mathrm{Ch}(S_A)$.

PROOF. In view of (4) we need only show $\mathrm{Ch}(S_A) \subset \mathrm{Ch}(A)$. Let k be the codimension of L_A in S_A. We may assume $k > 0$, and choose $g_1, \cdots, g_k$ in A^{-1} such that L_A and the $\log|g_i|$ span S_A. Set

$$M = \max_{1 \leqq i \leqq k} \|\log|g_i|\|, \qquad \alpha = \exp(-(2 + 2Mk)),$$

$$\beta = \exp(-(1 + 2Mk)),$$

so $0 < \alpha < \beta < 1$.

Suppose $x \in \mathrm{Ch}(S_A)$ and U is an open subset of X containing x. By [**B**, 2.2.6] there is $u \in S_A$ with $u < 0$, $u(x) > -1$ and $u < -(2+2Mk)$ on $X \backslash U$. We may write $u = v + \sum_{i=1}^k c_i \log|g_i|$ for some $v \in L_A$ and real numbers c_i; modifying u slightly if necessary, we may assume $v \in \log|A^{-1}|$. Write $c_i = [c_i] + \theta_i$, $0 \leqq \theta_i < 1$. Then

$$v + \sum_{i=1}^k [c_i] \log|g_i| - Mk = \log|g| \qquad \text{for some } g \in A^{-1},$$

and

$$\|\log|g| - (u - Mk)\| = \left\|\sum_{i=1}^{k} \theta_i \log|g_i|\right\| < Mk,$$

so $\log|g| < 0$, $\log|g(x)| > -(1+2Mk)$, and $\log|g| < -(2+2Mk)$ on $X\backslash U$. That is, $\|g\| < 1$, $|g(x)| > \beta$, and $|g| < \alpha$ on $X\backslash U$. This is the Bishop criterion [**B**, 2.3.4] that $x \in \mathrm{Ch}(A)$. $\square$

REMARK 5. Note that only the trivial implication (iii)$\Rightarrow$(ii) of [**B**, 2.3.4] is required here, and that [**B**, 2.2.6] is not needed at all in the hypo-Dirichlet case.

REMARK 6. Ted Gamelin has suggested that both Lemma 2 and the theorem would remain true if L_A were replaced by the (larger, by (3)) space $S_A \cap U_A$ where U_A is the space of $u \in C_R(X)$ such that $\int u\, dm_1 = \int u\, dm_2$ whenever m_1 and m_2 are in $\mathfrak{M}_x(A)$ for some $x \in X$. This assertion is correct, but is perhaps not a major improvement in general, since U_A may not be easy to identify.

REMARK 7. If γ is any nontrivial multiplicative linear functional on A, the sets $\mathfrak{M}_\gamma(S_A)$, $\mathfrak{M}_\gamma(L_A)$, $\mathfrak{M}_\gamma(\mathrm{Re}\, A)$ and $\mathfrak{M}_\gamma(A)$ can still be defined; $\mathfrak{M}_\gamma(\mathrm{Re}\, A)$ and $\mathfrak{M}_\gamma(A)$ are obvious, $\mathfrak{M}_\gamma(S_A)$ is the set of Arens-Singer measures for γ, and $\mathfrak{M}_\gamma(L_A)$ is the set of measures in $\mathfrak{M}_\gamma(A)$ which agree on L_A with (one of, hence all of) the measures in $\mathfrak{M}_\gamma(S_A)$. The relation (3) still holds as before, but the inclusion is often proper even if A is hypo-Dirichlet on X. In view of the inclusion $\mathfrak{M}_\gamma(A) \subset \mathfrak{M}_\gamma(L_A)$, the entire "local" theory for hypo-Dirichlet algebras (as developed, for instance, in [**AS**], [**G**], [**GL**], [**O**] and [**OW**]) holds intact whenever L_A has finite codimension in $C_R(X)$ and the latter is S_A.

That (4) was obtained from (3) hints that Lemma 2 might be proved by showing that $\mathfrak{M}_x(A) \subset \mathfrak{M}_x(S_A)$ for all $x \in X$. But Remark 5 already suggests that this is false (though true in the hypo-Dirichlet case), and I shall present a "minimal" example in which the hypotheses of the theorem hold but $\mathfrak{M}_x(A) \neq \mathfrak{M}_x(S_A)$ for a certain $x \in X$.

EXAMPLE. Let Y denote an open annulus in $\mathbf{C}$, $\overline{Y}$ its closure and X its boundary. Let $\hat{B} = \{\hat{f} \in C(\overline{Y}) : \hat{f}|\, Y$ is holomorphic$\}$ and $B = \hat{B}|X$. $\hat{B}$ and B are isometrically isomorphic under the correspondence $\hat{B} \ni \hat{f} \leftrightarrow f = \hat{f}|X \in B$. It is well known [**B**, p. 116] that B is hypo-Dirichlet; in fact, $S_B = C_R(X)$ while $(\mathrm{Re}\, B)^- = L_B$ has codimension 1 in $C_R(X)$. Fix $x \in X$ and $y \in Y$ and let $\hat{A} = \{\hat{f} \in \hat{B} : \hat{f}(x) = \hat{f}(y)\}$ and $A = \hat{A}|X$, a uniform algebra of (complex) codimension 1 in B. By Lemma 3 below, $(\mathrm{Re}\, A)^- = L_A$ has codimension 2 in $C_R(X)$ and S_A has codimension 1.

Let X_1 and X_2 be the two circles making up X and let m_i be normalized Lesbesgue measure on X_i, regarded as a measure on X; set $m=m_1+m_2$. If m_y is the (unique) Arens-Singer measure for y on B, it is known that m_y is mutually boundedly absolutely continuous with respect to m. Further, m_1-m_2 annihilates B. Thus if $\epsilon\neq 0$ is small enough, $\epsilon(m_1-m_2)$ and δ_x-m_y are linearly independent differences of measures in $\mathfrak{M}_x(A)$, and so $\mathfrak{M}_x(A)$ is 2-dimensional. On the other hand, δ_x and m_y are Arens-Singer measures for x on A, so $\mathfrak{M}_x(S_A)$ is 1-dimensional (a fact which can also be deduced from Lemma 2). Therefore $\mathfrak{M}_x(S_A)\subsetneq\mathfrak{M}_x(A)$.

Evidently $x\notin\mathrm{Ch}(A)$, so by the theorem or by direct verification, $\mathrm{Ch}(A)=X\backslash\{x\}$ and X is the Šilov boundary for A. □

We need only (5) and (7) in the technical lemma below. Though (6) nontrivially lengthens the proof, its inclusion here seems appropriate.

LEMMA 3. *Let B be a uniform algebra on X. Let γ and φ be distinct nontrivial multiplicative linear functionals on B, not both of which are evaluations at points of X. Evidently $A=\{f\in B:\gamma(f)=\varphi(f)\}$ is again a uniform algebra on X. Suppose there is $F\in B$ such that $\gamma(F)=\|F\|=1$ and $|\varphi(F)|<1$. Then*

(a) *For each $\epsilon>0$ and $a\in \mathbf{R}$ there is $g\in B$ such that $\gamma(G)=0$, $\varphi(G)=ia$ and $\|\mathrm{Re}\ G\|<\epsilon$.*

(b) *For any $g\in B$ such that* $\mathrm{Re}\,\gamma(g)\neq\mathrm{Re}\,\varphi(g)$,

$$(\mathrm{Re}\ B)^- = (\mathrm{Re}\ A)^- \oplus [\mathrm{Re}\ g], \tag{5}$$

$$L_B = L_A \oplus [\mathrm{Re}\ g], \tag{6}$$

$$S_B = S_A \oplus [\mathrm{Re}\ g], \tag{7}$$

$$(\log|B^{-1}|)^- = (\log|A^{-1}|)^- \oplus [\mathrm{Re}\ g]. \tag{8}$$

PROOF. Here $[\mathrm{Re}\,g]$ denotes the span of $\mathrm{Re}\,g$ in $C_R(X)$.

In (a), we may assume $a>0$. Let $Y=\{z\in\mathbf{C}:|z|<1\}$, $Z=\{z\in\mathbf{C}:|\mathrm{Re}\ z|<\epsilon/2 \text{ and } 0<\mathrm{Im}\ z<a+1\}$. Let g_1 be a homeomorphism of $\overline{Y}$ onto $\overline{Z}$ holomorphic on Y such that $g_1(1)=0$; the Riemann mapping theorem yields such a g_1. Let g_2 be an automorphism of $\overline{Y}$ such that $g_2(1)=1$ and $g_2(\varphi(F))=g_1^{-1}(ia)$. Then $G=g_1\circ g_2\circ F$ works.

In (b), we may assume for simplicity that $\gamma(g)=0$ and $\mathrm{Re}\ \varphi(g)=1$. If μ and ν are Arens-Singer measures for γ and φ respectively on the algebra B [**B**, 2.5.1] then $\mu-\nu$ annihilates S_A but not $\mathrm{Re}\ g$, so the sums in (5), (6), (7), (8) are indeed direct sums. Also, the left-hand sides of these equations clearly contain the right-hand sides.

Suppose $v\in\mathrm{Re}\ B$, say $v=\mathrm{Re}\,f$ for $f\in B$. Set $c=\mathrm{Re}\ \varphi(f)-\mathrm{Re}\ \gamma(f)$ and $h=f-cg-\gamma(f)\in B$. Then $\gamma(h)=0$ and $\varphi(h)=ia$ where $a\in\mathbf{R}$. By (a) there is a sequence $\{g_n\}$ in B with $\gamma(g_n)=0$, $\varphi(g_n)=ia$ and

$\|\text{Re } g_n\| \to 0$. Then $h - g_n + \gamma(f) \in A$, and if $h_n = \text{Re}(h - g_n + \gamma(f)) \in \text{Re } A$ then $v - c \text{ Re } g = h_n + \text{Re } g_n$. Letting $n \to \infty$ see that $v - c \text{ Re } g \in (\text{Re } A)^-$. This proves (5).

Next, let $v \in \log|B^{-1}|$. Write $v = \log|f|$, $f \in B^{-1}$. Set $c = \log|\varphi(f)/\gamma(f)|$ and $h = (\gamma(f))^{-1} f \cdot \exp(-cg) \in B^{-1}$, so $\gamma(h) = 1$ and $\varphi(h) = e^{ia}$ for some $a \in \boldsymbol{R}$. By (a) there is a sequence $\{g_n\}$ in B with $\gamma(g_n) = 0$, $\varphi(g_n) = ia$ and $\|\text{Re } g_n\| \to 0$. Note that $\gamma(f)h \cdot \exp(-g_n) \in A \cap B^{-1} = A^{-1}$, and set $h_n = \log|\gamma(f)h \cdot \exp(-g_n)| \in \log|A^{-1}|$. Then $v - c \text{ Re } g = h_n + \text{Re } g_n$. Letting $n \to \infty$ we obtain $v - c \text{ Re } g \in (\log|A^{-1}|)^-$. Now (7) and (8) follow.

By (7) there are unique functions $P: S_B \to S_A$ and $Q: S_B \to \boldsymbol{R}$ such that $v = P(v) + Q(v) \text{ Re } g$ for each $v \in S_B$. Clearly P and Q are linear, and by the open mapping theorem they are continuous.

Let $v \in L_B \cap S_A$. Fix a real number t. Since $tv \in L_B$ there is a sequence $\{v_n\}$ in $\log|B^{-1}|$ such that $\|v_n - tv\| \to 0$. Since $v_n = P(v_n) + Q(v_n) \text{ Re } g$, also $P(v_n) \in \log|B^{-1}|$. Since also $P(v_n) \in S_A$, by (7) and (8) we have $P(v_n) \in (\log|A^{-1}|)^-$. But $tv \in S_A$, so

$$\|P(v_n) - tv\| = \|P(v_n - tv)\| \leqq \|P\| \cdot \|v_n - tv\| \to 0.$$

Thus $tv \in (\log|A^{-1}|)^-$. Since this holds for each $t \in \boldsymbol{R}$ we have $v \in L_A$. Thus $L_B \cap S_A \subset L_A$, which by (7) is enough to prove (6).

Remark 8. Lemma 3 actually holds if only γ and φ lie in distinct Gleason parts for B; the proof of (a) must then be slightly modified. This fact, together with a characterization of subalgebras of finite codimension due to Gamelin [**G**, 9.8] and a fact about derivations [**C**], makes it possible to predict the codimension of $(\text{Re } A)^-$ in $(\text{Re } B)^-$ when A and B are uniform algebras on X and A is a subalgebra of B of finite codimension.

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