CONTRACTIBILITY OF THE AUTOMORPHISM GROUP OF A NONATOMIC MEASURE SPACE

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ABSTRACT. It is shown that the group of automorphisms of a nonatomic finite measure space is contractible.

1. In [1] S. Harada showed that the automorphism group of the unit interval with Lebesgue measure is simple and arcwise connected. Our purpose here is to give a simple proof that the automorphism group G of a nonatomic measure space $(\Omega, \mathcal{B}, m)$ with $m(\Omega) < \infty$ is contractible. The contraction map will be constructed directly by using induced transformations. Let B be the measure algebra of $(\Omega, \mathcal{B}, m)$; B is a metric space with the metric $|E, F| = m(E \triangle F)$ $(E, F \in B)$. An automorphism $T \in G$ maps B onto B isometrically. We shall consider both the *strong topology* on G (induced by uniform convergence on B) and the *weak topology* on G (induced by pointwise convergence on B). With either of these topologies, G is a topological group.

2. For $E \in B$ and $T \in G$ we set

$$\begin{aligned} &\Omega_0 = E^c, \\ (1) \qquad &\Omega_1 = E \cap T^{-1}E, \\ &\Omega_k = E \cap T^{-1}E^c \cap \cdots \cap T^{-(k-1)}E^c \cap T^{-k}E \qquad (k \geqq 2). \end{aligned}$$

By Poincaré's recurrence theorem, $\{\Omega_0, \Omega_1, \cdots\}$ forms a partition of Ω called the *return partition* of E and T. Now we define the *automorphism* T_E *induced by* E *and* T *by*

$$T_E(\omega) = T^k(\omega) \qquad (\omega \in \Omega_k,\ k = 0, 1, \cdots).$$

It is not hard to show that $T_E \in G$.

3. LEMMA. *The map* $\varphi: B \times G \to G$ *defined by* $\varphi(E, T) = T_E$ *is* (*weakly or strongly*) *continuous.*

PROOF. Let $F \in B$ and $(E^*, T^*) \in B \times G$ be fixed. Choose $\epsilon > 0$. Then there exists a positive integer k_0 such that

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$$m\left(\sum_{k=1}^{k_0} \Omega_k^*\right) > 1 - \epsilon,$$

where $\{\Omega_k^*\}$ is the return partition of E^* and T^*, and "$\sum$" denotes disjoint union. Because of the expressions (1), Ω_k depends continuously on the pair (E, T) for each k. There exists therefore a weak neighborhood N of (E^*, T^*) such that $(E, T) \in N$ implies

$$m\left(\sum_{k=1}^{k_0} \Lambda_k\right) > 1 - \epsilon,$$

where

$$\Lambda_k := \Omega_k^* \cap \Omega_k \qquad (1 \leqq k \leqq k_0).$$

By making the neighborhood N a bit smaller we may also assume that for each k with $1 \leqq k \leqq k_0$ and each $(E, T) \in N$

$$m(T^k(F \cap \Omega_k^*) \triangle T^{*k}(F \cap \Omega_k^*)) < \epsilon/k_0 \tag{2}$$

(note that $F \cap \Omega_k^*$ does not depend on (E, T)).

Then the following estimate is valid:

$$\begin{aligned} m(T_E(F) \triangle T_{E^*}^*(F)) &\leqq m\left(T_E\left(F \cap \sum_{k=1}^{k_0} \Lambda_k\right) \triangle T_{E^*}^*\left(F \cap \sum_{k=1}^{k_0} \Lambda_k\right)\right) + 2\epsilon \\ &\leqq 2\epsilon + \sum_{k=1}^{k_0} m(T^k(F \cap \Lambda_k) \triangle T^{*k}(F \cap \Lambda_k)). \end{aligned}$$

Now

$$\begin{aligned} m(T^k(F \cap \Lambda_k) \triangle T^{*k}(F \cap \Lambda_k)) \leqq\ & m(T^k(F \cap \Lambda_k) \triangle T^k(F \cap \Omega_k^*)) \\ &+ m(T^k(F \cap \Omega_k^*) \triangle T^{*k}(F \cap \Omega_k^*)) \\ &+ m(T^{*k}(F \cap \Omega_k^*) \triangle T^{*k}(F \cap \Lambda_k)), \end{aligned}$$

so that by using (2) we may continue the estimate to obtain

$$\begin{aligned} m(T_E(F) \triangle T_{E^*}^*(F)) &\leqq 2\epsilon + k_0 \cdot \frac{\epsilon}{k_0} + 2 \sum_{k=1}^{k_0} m((F \cap \Lambda_k) \triangle (F \cap \Omega_k^*)), \\ &\leqq 3\epsilon + 2m\left(\sum_{k=1}^{k_0} \Lambda_k \triangle \sum_{k=1}^{k_0} \Omega_k^*\right) \leqq 5\epsilon. \end{aligned}$$

Thus the mapping φ is weakly continuous. Replacing N by a strong neighborhood, we may assume that (2) is valid uniformly in $F \in B$, and then the above estimate holds for all $F \in B$. Therefore φ is strongly continuous.

4. THEOREM. *G is contractible.*

PROOF. Let J be the closed unit interval. Since $(\Omega, \mathcal{B}, m)$ is nonatomic and $m(\Omega) < \infty$, there exists a continuous map $\psi: J \rightarrow B$ such that $\psi(0) = \emptyset$ and $\psi(1) = \Omega$. Now define $\theta: J \times G \rightarrow G$ by $\theta(t, T) = \varphi(\psi(t), T)$ for $(t, T) \in J \times G$. By §3, θ is continuous, and it is obvious that

$$\theta(0, T) = I, \qquad \theta(1, T) = T,$$

where $I \in G$ is the identity. Thus G is contractible.

5. In any strong neighborhood of the identity, there exist automorphisms having no nth roots, so that one-parameter subgroups cannot pass through such points. It is not hard to show the following remarks, and we omit the proofs.

REMARK 1. If G has the strong topology, then there exist no nontrivial continuous one-parameter subgroups of G.

REMARK 2. Let $(\Omega, \mathcal{B}, m)$ be the unit interval with Lebesgue measure and suppose that G carries the weak topology. Then the continuous one-parameter subgroups form a dense set in G.

REFERENCES

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