ON AN INEQUALITY OF T. J. WILLMORE

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ABSTRACT. Willmore proved that the integral of the square of mean curvature H over a closed surface M^2 in E^3, $\int_{M^2} H^2 dV$, is $\geqq 4\pi$, and equal to 4π when and only when M^2 is a sphere in E^3. In this paper we give some generalizations of Willmore's result.

Let $x: M^2 \to E^3$ be an immersion of an oriented closed surface M^2 into euclidean 3-space E^3. In [6], [7], Willmore proved the following inequality for the mean curvature $H(p)$ of M^2 in E^3:

$$\int_{M^2} H^2(p) dV \geqq 4\pi, \tag{1}$$

where the equality holds when and only when M^2 is imbedded as a sphere. The main aim of this paper is to give some generalizations of the inequality (1).

ACKNOWLEDGEMENT. The author would like to express his thanks to Professor T. Nagano for his kind help during the preparation of this paper.

1. **Preliminaries.** Let M^n be an n-dimensional oriented closed manifold with an immersion $x: M^n \to E^{n+N}$. Let $F(M^n)$ and $F(E^{n+N})$ be the bundles of oriented orthonormal frames of M^n and E^{n+N} respectively. Let B be the set of elements $b = (p, e_1, \cdots, e_{n+N})$ such that $(p, e_1, \cdots, e_n) \in F(M^n)$ and $(x(p), e_1, \cdots, e_{n+N}) \in F(E^{n+N})$ whose orientation is coherent with the one of E^{n+N}, identifying e_i with $dx(e_i)$, $i = 1, \cdots, n$. Then $B \to M^n$ may be considered as a principal bundle with fibre $SO(n) \times SO(N)$, and $\tilde{x}: B \to F(E^{n+N})$ is naturally defined by $\tilde{x}(b) = (x(p), e_1, \cdots, e_{n+N})$. Let B_v be the set of normal unit vectors of M^n in E^{n+N}, and $B_v \to M^n$ is a sphere bundle whose fibre at $p \in M^n$ is S_p^{N-1}. Let $\tilde{v}: B_v \to S_0^{n+N-1}$ be the mapping such that $\tilde{v}(p, e)$ is the unit vector at the origin of E^{n+N} and parallel to e.

The structure equations of E^{n+N} are given by

Received by the editors February 10, 1970.

AMS 1969 *subject classifications.* Primary 5301, 5342; Secondary 5372, 5375.

Key words and phrases. ith mean curvature, Lipschitz-Killing curvature, αth curvatures of first or second kinds, Willmore's inequality, betti number.

$$dx = \sum_A \theta_A e_A, \qquad de_A = \sum_B \theta_{AB} e_B,$$

$$(2)\quad d\theta_A = \sum_B \theta_B \wedge \theta_{BA}, \quad d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB}, \qquad \theta_{AB} + \theta_{BA} = 0,$$

$$A, B, C, \cdots = 1, 2, \cdots, n + N.$$

where θ_A, θ_{AB} are differential 1-forms on $F(E^{n+N})$. Let ω_A, ω_{AB} be the induced 1-forms on B from θ_A, θ_{AB} by the mapping $\tilde{x}$. Then we have

$$(3)\quad \begin{aligned} &\omega_r = 0, \quad \omega_{ri} = \sum_j A_{rij}\omega_j, \qquad A_{rij} = A_{rji}. \\ &i, j, k, \cdots = 1, 2, \cdots, n; \; r, s, t, \cdots = n + 1, \cdots, n + N. \end{aligned}$$

From (2) we get

$$(4)\qquad d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \qquad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}.$$

For any $(p, e_r) \in B_v$, we put

$$(5)\qquad \mathrm{I} = dx \cdot dx, \qquad \mathrm{II}_r = de_r \cdot dx.$$

The eigenvalues $k_1(p, e_r), \cdots, k_n(p, e_r)$ of II_r relative to I are called the principal curvatures of M^n associated with (p, e_r). The hth mean curvature $H_n(p, e_r)$ associated with (p, e_r) is defined by the following equation

$$(6)\qquad \det(\delta_{ij} + tA_{rij}) = \sum_{h=0}^{n} \binom{n}{h} H_h(p, e_r) t^h,$$

where δ_{ij} is the Kronecker delta. If there is no danger of confusion, we shall simply denote $k_i(p, e_r)$ and $H_i(p, e_r)$ by $k_i(p)$ (or k_i) and $H_i(p)$ (or H_i) respectively. It is easy to see that

$$(7)\qquad \binom{n}{h} H_h = \sum k_1 \cdots k_h, \qquad h = 1, 2, \cdots, n,$$

and $H_0 = 1$. Throughout this paper, we simply denote $H_n(p, e_r)$ by $K(p, e_r)$. $K(p, e_r)$ is called the Lipschitz-Killing curvature at (p, e_r).

2. **αth curvature of first and second kinds.** Let M^2 be a surface immersed in E^{2+N}. Let $(p, e_1, e_2, \bar{e}_3, \cdots, \bar{e}_{2+N})$ be a local cross section of $B \to F(M^2)$ and for any e in S_p^{N-1}, $p \in U$, put $e = e_{2+N} = \sum_r \xi_r \bar{e}_r(p)$. Denoting the restriction of A_{rij} on the image of this local cross section by $\overline{A}_{rij}$, we may put

$$A_{2+Nij} = \sum_r \xi_r \overline{A}_{rij}.$$

From (3) and (6), we get

$$K(p, e) = \det\left(\sum_r \xi_r \bar{A}_{rij}\right) \tag{8}$$
$$= \left(\sum_r \xi_r \bar{A}_{r11}\right)\left(\sum_s \xi_s \bar{A}_{s22}\right) - \left(\sum_t \xi_t \bar{A}_{t12}\right)^2.$$

The right-hand side is a quadratic form of $\xi_3, \cdots, \xi_{2+N}$. Hence, by choosing a suitable cross section, we can write $K(p, e)$ as

$$K(p, e) = \sum_r \lambda_{r-2}\xi_r\xi_r, \qquad \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_N. \tag{9}$$

We call this local cross section of $B \rightarrow F(M^2)$, the Frenet cross section in the sense of Ōtsuki, and the frame $(p, e_1, e_2, \bar{e}_3, \cdots, \bar{e}_{2+N})$ the Ōtsuki's frame [5]. We call the curvature λ_α, the αth curvature of the second kind. With respect to this Ōtsuki's frame the curvatures:

$$\mu_\alpha(p) = H_1(p, \bar{e}_{\alpha+2}) \tag{10}$$

are called the αth curvature of the first kind. By means of the method of definitions, μ_α, λ_α are defined continuous on the whole manifold M^2 and are differentiable on the open subset in which $\lambda_1 > \lambda_2 > \cdots > \lambda_N$.

With respect to the Ōtsuki frame, we have [5]

$$\omega_{1r} \wedge \omega_{2r} = \lambda_{r-2} dV, \qquad dV = \omega_1 \wedge \omega_2, \; r = 3, \cdots, 2 + N, \tag{11}$$

$$G(p) = \sum_{\alpha=1}^{N} \lambda_\alpha(p), \tag{12}$$

where $G(p)$ denotes the Gaussian curvature of M^2 in E^{2+N}, and we also have

$$H_1(p, e) = \sum_{\alpha=1}^{N} \cos\theta_\alpha \mu_\alpha(p), \qquad e = \sum_r \cos\theta_{r-2}\bar{e}_r. \tag{13}$$

As in [2], [3], [5], we know that the forms:

$$d\sigma = \omega_{2+N,3} \wedge \cdots \wedge \omega_{2+N,1+N}, \quad \text{and} \quad dV \wedge d\sigma \tag{14}$$

can be regarded as the volume elements of S_p^{N-1} and B_v respectively.

3. Some generalizations of Willmore's inequality.

THEOREM 1. *Let $x: M^2 \rightarrow E^{2+N}$ be an immersion of an oriented closed surface M^2 into E^{2+N}. Then the sum of the squares of the αth curvatures* *of the first kind satisfies the following inequality*:

$$\int_{M^2}\left(\sum_{\alpha=1}^{N}\lambda_\alpha^2\right)dV \geqq 4\pi, \tag{15}$$

where the equality holds when and only when M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N}.

PROOF. Let $(p, e_1, e_2, \bar{e}_3, \cdots, \bar{e}_{2+N})$ be an Ōtsuki's frame, then by (9) we know that the Lipschitz-Killing curvature $K(p, e)$ satisfies

$$\int_{B_v} K(p,e)dV\wedge d\sigma = \int_{B_v}(\lambda_1(p)\cos^2\theta_1 + \cdots + \lambda_N(p)\cos^2\theta_N)dV\wedge d\sigma$$
$$= \frac{c_{N+1}}{2\pi}\int_{M^2}\left(\sum_{\alpha=1}^{N}\lambda_\alpha^2(p)\right)dV = \frac{c_{N+1}}{2\pi}\int_{M^2} G(p)dV.$$

Thus by the Gauss-Bonnet formula, we have

$$\int_{B_v} K(p,e)dV\wedge d\sigma = (2-2g)c_{N+1}, \tag{16}$$

where g denotes the genus of M^2 and c_{N+1} denotes the volume of the unit $(N+1)$-sphere. On the other hand, by an inequality of Chern-Lashof [**3**], we have

$$\int_{B_v} |K(p,e)|\, dV\wedge d\sigma \geqq (2+2g)c_{N+1}. \tag{17}$$

Therefore, if we set

$$V+ = \{(p,e)\in B_v : K(p,e)\geqq 0\}, \qquad V- = \{(p,e)\in B_v : K(p,e) < 0\}, \tag{18}$$

then, by (16) and (17), we get

$$\int_{V+} K(p,e)dV\wedge d\sigma \geqq 2c_{N+1}. \tag{19}$$

It is easy to see that the equality of (19) holds when and only when

$$\int_{B_v} |K(p,e)|\, dV\wedge d\sigma = (2+2g)c_{N+1}. \tag{20}$$

Now, we have the following identity:

$$(k_1+k_2)^2 = (k_1-k_2)^2 + 4k_1k_2.$$

Hence we get

$$\begin{aligned}(21)\qquad \int_{B_v} (H_1(p, e)^2)dV \wedge d\sigma &\geqq \int_{V+} (H_1(p, e)^2)dV \wedge d\sigma \\ &\geqq \int_{V+} K(p, e)dV \wedge d\sigma.\end{aligned}$$

Therefore, by (19), we get

$$(22)\qquad \int_{B_v} (H_1(p, e))^2 dV \wedge d\sigma \geqq 2c_{N+1}.$$

On the other hand, by (14) and the following formulas:

$$\begin{aligned}\int_{S_p^{N-1}} \cos\theta_\alpha \cos\theta_\beta \, d\sigma &= c_{N+1}/2, \quad \text{if } \alpha = \beta, \\ &= 0, \qquad\quad \text{if } \alpha \neq \beta,\end{aligned}$$

we have

$$\begin{aligned}(23)\qquad \int_{B_v} (H_1(p, e)^2)dV \wedge d\sigma &= \sum_{\alpha,\beta=1}^{N} \int_{B_v} \mu_\alpha(p)\mu_\beta(p) \cos\theta_\alpha \cos\theta_\beta \, dV \wedge d\sigma \\ &= \frac{c_{N+1}}{2\pi} \int_{M^2} \left(\sum_\alpha \mu_\alpha^2(p) \right) dV.\end{aligned}$$

Hence, by (22) and (23), we get (15). Now, suppose that the equality of (15) holds, then, by (19), (20) and (21), we get

$$(24)\qquad H_1(p, e)^2 = K(p, e), \quad \text{and} \quad k_1(p, e) = k_2(p, e),$$

for all (p, e) in B_v. Thus, by (9) and (24), we know that the last curvature of the second kind $\lambda_N \geqq 0$, for all p in M^2. Hence, by Lemma 1 of [2], we know that M^2 is imbedded as a convex surface in a 3-dimensional linear subspace, say E^3, of E^{2+N}. Thus, by (24), we know that M^2 is imbedded as a sphere in E^3. Conversely, if M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N}, then it is easy to see that the equality of (15) holds. This completes the proof of the theorem.

In the following, we assume that $x: M^{2m} \to E^{2m+1}$ is an immersion of an oriented even-dimensional closed manifold M^{2m} into E^{2m+1}. Let $\bar{e}$ denote the outer normal vector on M^{2m} in E^{2m+1}. Set

$$(25)\qquad g(p) = H_m(p)^2 - K(p),$$

where $H_m(p) = H_m(p, \bar{e})$ and, $K(p) = K(p, \bar{e})$. By Theorem 1 of [1], we have the following proposition. We omit the proof.

PROPOSITION 2. *Let $x: M^{2m} \to E^{2m+1}$ be an immersion of an oriented $2m$-dimensional closed manifold M^{2m} in E^{2m+1} with the following property*:

(A) $$\{p \in M^{2m}: g(p) \geqq 0\} \supseteq \{p \in M^{2m}: K(p) \geqq 0\}.$$

Then we have the following inequality:

(26) $$\int_{M^{2m}} H_m(p)^2 dV \geqq \tfrac{1}{2} c_{2m} \sum_i \beta_{2i},$$

where β_i denotes the ith betti number of M^{2m}. The equality of (26) *holds when and only when the immersion x is a minimal imbedding* [1].

REMARK. If $m=1$, then (A) always holds.

THEOREM 3. *Let $x: M^{2m} \to E^{2m+1}$ be an immersion of an oriented $2m$-dimensional closed manifold M^{2m} in E^{2m+1} with nonnegative principal curvatures. Then the mth mean curvature H_m satisfies the following*:

(27) $$\int_{M^{2m}} (H_m(p)^2) dV \geqq \tfrac{1}{2} c_{2m} \sum_i \beta_i,$$

where the equality holds when and only when M^{2m} is imbedded as a sphere.

PROOF. By the assumption, $k_1, \cdots, k_n \geqq 0$, we have [4, pp. 104–105]

(28) $$H_m^2 \geqq K \geqq 0,$$

where the equality holds when and only when $k_1 = k_2 = \cdots = k_n$. Thus, by Proposition 2, we have (26). Furthermore, by Theorem 3 of [1], we know that all odd-dimensional betti numbers of M^{2m} vanish. Hence, by (26), we get (27). Now, suppose that the equality of (27) holds. Then we get $H_m^2(p) = K(p)$ for all $p \in M^{2m}$. Hence, we get $k_1(p) = \cdots = k_n(p)$ for all $p \in M^{2m}$. Thus M^{2m} is imbedded as a sphere in E^{2m+1}. This completes the proof of the theorem.

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