A CHARACTERIZATION OF TOTAL GRAPHS

M. BEHZAD

I. ABSTRACT. We consider "ordinary" graphs; that is, finite undirected graphs with no loops or multiple edges. The *total graph* $T(G)$ of a graph G is that graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if they are adjacent or incident in G. A characterization of regular total graphs as well as some other properties of total graphs have been considered before. In this article we consider nonregular graphs and yield a method which enables us actually to determine whether or not they are total.

II. **Introduction.** We consider (ordinary) graphs; that is, finite undirected graphs with no loops or multiple edges. Besides the chromatic number $\chi(G)$ and edge chromatic number $\chi_1(G)$ there is associated with G another positive integer $\chi_2(G)$, called the *total chromatic number* of G, which is the minimum number of colors required for coloring the elements (edges and vertices) of G such that no two elements which are either adjacent or incident have the same color. (The Total Chromatic Conjecture [3] states that $\chi_2(G) \leqq 2 + \max \deg G$. In this conjecture G can be replaced by multigraphs M containing no 4-regular multigraph of order 3 as a subgraph.[1]) The total graph $T(G)$ of G is defined in such a way that $\chi_2(G) = \chi(T(G))$—in analogy with the well-known formula $\chi_1(G) = \chi(L(G))$, where $L(G)$ is the line graph of G. The *total graph* $T(G)$ of G is that graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in G. For an illustration a graph G is given in Figure 1 together with $L(G)$ and $T(G)$. Both G and $L(G)$ are disjoint induced subgraphs of $T(G)$.

A characterization of regular total graphs is given in [7] and some other properties of total graphs are considered in [1], [2], [4]. In this article we yield a method which enables us actually to determine whether or not any given graph is total.

III. **Results.** In [6] it was proved that $G_1 = G_2$-isomorphism is denoted by the equality sign—if and only if $T(G_1) = T(G_2)$. There, it

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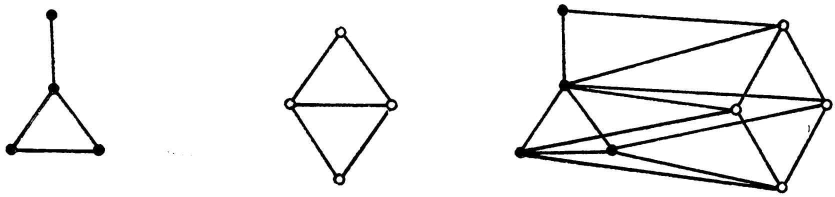
[1] This restriction on M is necessary as was observed by the author, E. Jacovič and others.



G $L(G)$ $T(G)$

FIGURE 1

was also shown that, apart from isomorphism, G is the only subgraph of $H=T(G)$ whose total graph is H, where G is a connected graph which is neither a cycle C nor a complete graph K. (Definitions not given here may be found in [5], [8].) This subgraph called *the special subgraph* of H is denoted by G_s. The subgraph G_s is induced by a set of vertices of H which are called the *special vertices* of H; any other element of $V(H)$ is a *nonspecial element* of $V(H)$. The subgraph induced by the set of nonspecial vertices of H is $L(G_s)$ and in H each nonspecial vertex is adjacent with exactly two adjacent special elements.

Suppose that H is a connected total graph which is neither $T(C)$ nor $T(K)$, and that $v\in V(H)$ is nonspecial. Let $\{i\}$, $i=0, 1, 2, \cdots, n$, denote the class of all vertices of H whose distance from v is i. Our hypotheses imply that $n\geqq 2$. Then we have the following theorem.

THEOREM 1. *Let H, $H\neq T(C)$, $T(K)$, be a connected total graph and let v be a nonspecial element of $V(H)$. Then each nonspecial vertex of H in $\{i\}$, $i\geqq 1$, is adjacent with exactly two special vertices of H both of which are in $\{i\}$, or one is in $\{i\}$ and the other in $\{i+1\}$.*

PROOF. We use induction on i. Let the two adjacent special vertices of H which are adjacent with v be denoted by u_1 and v_1. It is clear that u_1 and v_1 are the only special elements of $V(H)$ which are in $\{1\}$; and that each nonspecial element of $\{1\}$ is adjacent with two special vertices of H one of which is in $\{2\}$ and the other in $\{1\}$.

Let w be a nonspecial vertex of H in $\{2\}$. If w is adjacent with u_1 or v_1, then w and v are adjacent in H and $w\in\{1\}$ which is a contradiction. Clearly w is adjacent with no element of $\{i\}$, $i\geqq 4$. Next, we show that w is not adjacent with two special elements of $\{3\}$. Assume this is the case. Then the vertex w, which is adjacent neither with u_1 nor with v_1, is adjacent with a nonspecial element w_1 of $\{1\}$. But this is impossible, since the preimages of w and w_1 under the total graph function have no vertices in common. Hence the assertion follows for $\{2\}$.

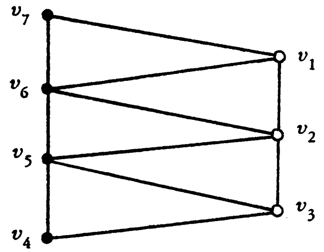


FIGURE 2

Assume the assertion is true for i, $2 \leqq i < n$, and let w be a nonspecial vertex of H in $\{i+1\}$. First, we show that w is not adjacent with two special elements of $\{i+2\}$. Assume this is the case. The vertex w is adjacent with an element w_i of $\{i\}$. The vertex w_i is not special since otherwise w is adjacent with three or more special vertices of H. w_1 is not nonspecial either, because otherwise the induction hypothesis would imply that the vertices w and w_i are not adjacent.

Next, we show that w is not adjacent with any special vertex of H in $\{i\}$. Assume this is not the case, and let u_1 be a special vertex of H adjacent with w which is in $\{i\}$. The vertex u_i is adjacent with a vertex u of $\{i-1\}$. If u is special, then by the induction hypothesis the image of the edge uu_i under the total graph function, say u_{i-1}, is in $\{i-1\}$, and the vertices w and u_{i-1} are adjacent. Thus $w \notin \{i+1\}$ which contradicts our assumption. Hence u is a nonspecial vertex of $\{i-1\}$. Then u is adjacent with a special element of $\{i-1\}$, say u'. In this case the image of the edge u_iu' is u. Hence u and w are adjacent and it follows that $w \notin \{i+1\}$. This contradiction proves our assertion and completes the proof of the theorem.

The following corollaries are of value to us.

COROLLARY 1. *Under the assumptions of Theorem* 1, *no nonspecial element of* $\{i\}$ *is adjacent with a special element of* $\{i-1\}$, $i \geqq 2$.

COROLLARY 2. *Under the assumptions of Theorem* 1, *every class* $\{i\}$, $1 \leqq i \leqq n-1$, *contains both special and nonspecial vertices of* H; *while the class* $\{n\}$ *cannot solely contain some nonspecial vertices of* H.

PROOF. Assume $H = T(G)$. Then G and $L(G)$ are two disjoint connected subgraphs of H. Hence if $\{i\}$, $1 \leqq i \leqq n-1$, solely contains nonspecial (resp. special) vertices of H, then the class $\{j\}$, $j \geqq i+1$, cannot contain any special (resp. nonspecial) element of $V(H)$. This observation together with Theorem 1 imply that no class $\{i\}$, 1

$\leqq i \leqq n$, can entirely contain some nonspecial vertices of H. Next, we show that no class $\{i\}$, $1 \leqq i \leqq n-1$, can contain only special elements. Assume so, and let $u \in \{i\}$ for which there exists a special element w in $\{i+1\}$ adjacent with u. Then, under the total graph function the edge uw has no image. This contradiction proves the corollary.

In Figure 2 we present a graph H which is total and in which the class $\{n\} = \{3\}$ contains a special element alone. (We note that G_s is the path v_4, v_5, v_6, v_7. Hence the vertices v_1, v_2 and v_3 are the nonspecial vertices of H.) The classes $\{i\}$, $i=0, 1, 2, 3$, are determined with respect to the vertex v_1 and they are: $\{0\} = \{v_1\}$, $\{1\} = \{v_2, v_6, v_7\}$, $\{2\} = \{v_3, v_5\}$, and $\{3\} = \{v_4\}$.

COROLLARY 3. *Under the assumptions of Theorem* 1, *each nonspecial element in* $\{i\}$ *is adjacent with at least one nonspecial element in* $\{i-1\}$, *for* $i \geqq 1$, *and each special element in* $\{i\}$, $i \geqq 2$, *is adjacent with at least one special element in* $\{i-1\}$.

PROOF. This follows directly from Theorem 1, and the fact that both G_s and $L(G_s)$ are two connected subgraphs of H.

THEOREM 2. *Assume* H, $H \neq T(C)$, $T(K)$, *is a connected total graph and that* v *is a nonspecial element of* $V(H)$ *adjacent with the special vertices* v_1 *and* u_1 *of* H. *Then we can determine the subgraph* G_s *completely.*

PROOF. Since G_s is an induced subgraph of H, it suffices to determine the set of special vertices of H. We do this by separating the sets of special and the sets of nonspecial elements S_i and N_i, respectively, of each class $\{i\}$, $i=0, 1, 2, \cdots, n$, formed with respect to the vertex v.

It is clear that the class $\{0\}$ contains no special element. Thus $S_0 = \varnothing$, and $N_0 = \{0\} - S_0 = \{0\} = \{v\}$. The class $\{1\}$ contains exactly two special elements and they are the vertices u_1 and v_1. Therefore $S_1 = \{u_1, v_1\}$, and $N_1 = \{1\} - S_1$. Clearly $N_1 \neq \varnothing$.

Let $w_1 \in N_1$. Then by the proof of Theorem 1, w_1 is adjacent with one of u_1 and v_1, say u_1, and a special element of H, say u_2, which is in $\{2\}$. This vertex u_2 which we propose to determine is necessarily adjacent with u_1. Since no nonspecial element of $\{2\}$ is adjacent with a special element of $\{1\}$ any element of $\{2\}$ adjacent with u_1 is necessarily special. Among these only one is adjacent with both u_1 and w_1 since otherwise the vertex w_1 will be adjacent with three or more special vertices of H. Thus the vertex u_2 can uniquely be determined. We repeat this argument for all elements of N_1 and in this manner we obtain a set S_2' consisting of some special elements of $\{2\}$. We observe that $S_2' \neq \emptyset$ since $N_1 \neq \emptyset$. Let $w_2 \in \{2\} - S_2'$. The vertex w_2 is a nonspecial element of H. For otherwise, by Corollary 3, the vertex w_2 is adjacent with a special element, say v_1', of $\{1\}$ and the edge w_2v_1' must correspond, under the total graph function, to a nonspecial element, say w_1', of H. But the vertex w_1' can only be in the class $\{1\}$. This contradicts the fact that w_2 is an element of $\{2\} - S_2'$. Thus $N_2 = \{2\} - S_2'$ and $S_2 = S_2'$.

Next, we separate special and nonspecial elements of $\{3\}$. Let $w_2 \in N_2$. If w_2 is adjacent with two elements of S_2, then each element of $\{3\}$ adjacent with w_2 is a nonspecial element of $\{3\}$; otherwise w_2 is adjacent with a special element, say u_2, of $\{2\}$ and a special element, say u_3, of $\{3\}$. We propose to determine u_3. Again, as was seen before, all vertices of $\{3\}$ adjacent with u_2 are special vertices of $\{3\}$ among which only one is adjacent with both w_2 and u_2. This vertex is the vertex u_3 which we are looking for. We repeat this argument for every element of N_2 and obtain a set S_3' consisting of some special elements of $\{3\}$. We observe that every element of $\{3\} - S_3'$ is nonspecial. (The proof is similar to the one given for $\{2\} - S_2'$.) Thus $N_3 = \{3\} - S_3'$, and $S_3 = S_3'$.

Using induction and the above procedure we separate the special and nonspecial elements of $\{i\}$, $i = 0, 1, 2, \cdots, n$. Now we let $S = S_0 \cup S_1 \cup \cdots \cup S_n$. Then $G_s = \langle S \rangle$, and $H = T(G_s)$. This completes the proof of the theorem.

Since in the above theorem the vertices v, u_1, and v_1 play an important role for the determination of G_s, we denote G_s by $G_{v;u_1v_1}$.

Let u be an arbitrary vertex of a graph H. We denote the set consisting of u and all vertices of H adjacent with u by $\overline{N}(u)$; this set is called the *closed neighborhood* of u. Now we are prepared to present the main theorem of this article.

THEOREM 3. *Assume that H, $H \neq T(C)$, $T(K)$, is a connected graph, and that u is an arbitrary vertex of H. Then H is total if and only if $H = T(G_{v;u_1v_1})$ for some $v \in \overline{N}(u)$ and some edge u_1v_1, where u_1 and v_1 are two even vertices of H adjacent with v.*

PROOF. Assume that H is total. Then some element v of $\overline{N}(u)$ is a nonspecial vertex of H. Under the total graph function, v corresponds to an edge u_1v_1 of the special subgraph of H. The vertices u_1 and v_1 have even degrees in H and by the definition of total graphs both u_1 and v_1 are adjacent with v. Thus the special subgraph of H is $G_{v;u_1v_1}$ and $H = T(G_{v;u_1v_1})$. The converse is trivial.

Theorems 2 and 3 answer the main question mentioned at the end of the introduction, namely, when is H total. The answer is that we try appropriate v, u_1, v_1, find $G_{v;u_1v_1}$ by the algorithm provided in the proof of Theorem 2, and then see if $T(G_{v;u_1v_1})$ is H.

The characterization of disconnected total graphs is reduced to that of connected ones since if G has m components, then $T(G)$ consists of m components each of which is the total graph of a component of G. The converse is also true. Thus to completely characterize total graphs it remains to give a characterization of $T(C)$ and $T(K)$. The following results are obtained in [7] and for completeness we state them next.

(i) Let G be a connected regular graph of degree 4. The only such graph with fewer than seven vertices is $T(K_3)$ which is the total graph of each of its triangles. If $|V(G)| \geqq 7$, then G is total if and only if: (a) $|V(G)| = 2n$, n a positive integer, (b) $V(G)$ is the disjoint union of two sets each inducing a cycle C of order n, and (c) G is the total graph of C.

(ii) Let v be a vertex of a graph H, and let G be a maximal complete graph contained in the subgraph induced by $\overline{N}(v)$. Then $H = T(K)$ if and only if $H = T(G)$.

The procedure given in the proof of Theorem 2 for determining whether or not a connected graph H, $H \neq T(C)$, $T(K)$, is total seems rather long. There are often easy ways to fix v or to eliminate many possibilities for u_1 and v_1. For example, if H has odd vertices, then v might be taken to be an odd vertex of H.

The following theorem is useful for the determination of those graphs which are total graphs of connected graphs having vertices of degree 1.

THEOREM 4. *Let H be a connected nonregular total graph with a vertex v_1 of degree 2 adjacent with vertices v and u_1 of H. Then:*

(a) $\deg u_1 \neq \deg v$, *say* $\deg v < \deg u_1$;

(b) *v is nonspecial while u_1 and v_1 are the special vertices of H having the property that v corresponds, under the total graph function, to the edge u_1v_1.*

PROOF. It is clear that the degree of each nonspecial vertex of H is greater than two. Thus v_1 is special. Hence either u_1 or v is nonspecial. Assume $\deg u_1 = \deg v$. Then $\deg u_1 = \frac{1}{2} \deg v + 1$, or $\deg v = \frac{1}{2} \deg u_1 + 1$, both implying that H is regular. Thus we may assume that $\deg v < \deg u_1$.

To prove part (b) suppose that u_1 is nonspecial. Then $\deg u_1 = \frac{1}{2} \deg v + 1$ and the inequality $\deg v < \deg u_1$ implies the impossible inequality deg $v_1 \leqq 1$. Hence v is nonspecial and clearly v corresponds to the edge u_1v_1.

Theorems 3 and 4 provide an easy and practical characterization of total graphs of trees.

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PAHLAVI UNIVERSITY, SHIRAZ, IRAN

INSTITUTE FOR MATHEMATICAL RESEARCH, THE IMPERIAL SCIENTIFIC RESEARCH ORGANIZATION, TEHRAN, IRAN