POSITIVE MATRIX $H^{1/2}$ AND HERMITIAN MATRIX H^1 FUNCTIONS ARE CONSTANT

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ABSTRACT. It is known that if $f \in H^{1/2}$ and $f(z) \geqq 0$ a.e. for $|z| = 1$ then $f(z)$ is a constant. Also if $f \in H^1$ and $f(z)$ is real a.e. for $|z| = 1$, then f is a constant. In this note we extend these two results to matrix-valued functions.

Introduction. In [3] J. Neuwirth and D. J. Newman proved that if $f(z) \in H^{1/2}$ and $f(z) \geqq 0$ a.e. for $|z| = 1$ then $f(z)$ is a constant. They also noted that this behavior is extreme in the sense that $z/(1+z)^2$ is positive everywhere on $|z| = 1$, $z \neq 1$, and lies in H^p for every $p < 1/2$. It is well known (cf. [1, Theorem 3.9]) that $f \in H^1$ if and only if $f(z) = (1/2\pi)\int_0^{2\pi} P(r, \theta - t)f(e^{it})dt$, $z = re^{i\theta}$. This implies that if $f \in H^1$ and $f(z)$ real a.e. for $|z| = 1$, then f is a constant. Note that this behavior is also extreme as $i(1+z)/(1-z)$ shows. We wish to extend these two results to matrix-valued H^p functions (cf. [2]).

THEOREM 1. *If $F(z) = [f_{kj}(z)]$, $1 \leqq k, j \leqq q$, is a $q \times q$ matrix-valued function in $H^{1/2}$ and $F(z) \geqq 0$ (positive definite) a.e. for $|z| = 1$, then $F(z)$ is a constant matrix.*

PROOF. $F(e^{i\theta})$ positive definite a.e. implies $f_{jj}(e^{i\theta}) \geqq 0$ a.e. for $j = 1, \cdots, q$. Thus each $f_{jj}(z)$ is a constant by [3]. Also, we have for each $k \neq j$ that

$$f_{kk}(e^{i\theta}) \cdot f_{jj}(e^{i\theta}) \geqq f_{kj}(e^{i\theta}) \cdot f_{jk}(e^{i\theta}) = f_{kj}(e^{i\theta})\overline{f_{kj}(e^{i\theta})} = |f_{kj}(e^{i\theta})|^2 \quad \text{a.e.}$$

Hence $f_{kj}(e^{i\theta}) \in L_1$ for all $k \neq j$ and

$$f_{kj}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)f_{kj}(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\overline{f_{jk}(e^{it})}dt = \overline{f_{jk}(z)}$$

implying that $f_{kj}(z)$ is a constant for all k and j.

The matrix-valued function $F(z) = [z/(1+z)^2 I]$ shows that the theorem fails to hold for $p < 1/2$ as in the scalar case.

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THEOREM 2. *If* $F(z) = [f_{kj}(z)]$, $1 \leqq k, j \leqq q$, *is a* $q \times q$ *matrix-valued function in* H^1 *and* $F(z) = F^*(z)$ *(hermitian) a.e. for* $|z| = 1$, *then* $F(z)$ *is a constant.*

PROOF. $F(e^{i\theta})$ hermitian a.e. implies $f_{jj}(e^{i\theta})$ real a.e. for $j = 1, \cdots, q$. Thus, as shown in the introduction, each $f_{jj}(z)$ is a constant. For each $k \neq j$, we have

$$f_{kj}(e^{i\theta}) = \overline{f_{jk}(e^{i\theta})}.$$

Since $f_{kj} \in L_1$, the same line of reasoning as given in the proof of Theorem 1 implies that $f_{kj}(z)$ is a constant.

The matrix-valued function $F(z) = [i(1+z)/(1-z)I]$ shows that this theorem also fails for $p < 1$.

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