DETERMINING THE CELLULARITY OF A 1-COMPLEX BY PROPERTIES OF ITS ARCS

R. B. SHER[1]

ABSTRACT. We show that a 1-complex K topologically embedded in the interior of a topological n-manifold M, $n \geq 3$, satisfies the cellularity criterion if for each arc A in K, $M-A$ is 1-LC at an endpoint of A. This condition is satisfied if each arc in K is LPU at an endpoint. An example is given to show that it is not sufficient to suppose that each arc in K satisfies the cellularity criterion.

1. **Introduction.** An important concept in the study of manifolds has been that of cellularity; in turn, one of the most useful tools in the study of cellularity has been the cellularity criterion of McMillan [**14**]. During the study of these notions, several results have been obtained which imply that if a continuum satisfies the cellularity criterion, then certain of its subcontinua also do. As an example, a result of McMillan [**15**, Theorem 1] implies that if K is a contractible 1-complex topologically embedded in the interior of an n-manifold, $n \geqq 3$, and K satisfies the cellularity criterion, then each subcomplex of K also does. In this paper we turn the problem about; that is, we seek properties of subcontinua which imply that a continuum satisfies the cellularity criterion.

The starting point for our investigations was a conjecture of Stewart [**17**, Conjecture 5] to the effect that an n-frame is cellular in E^3 if each arc in it is cellular. Its truth would provide a nice complement to the result of McMillan mentioned above. In §4 we show that the conjecture is false. In particular we construct, for $n \geqq 3$, an n-frame in E^3 each of whose $(n-1)$-frames is cellular but which is not itself cellular. The construction is similar to the construction of an arc given in [**1**] by Alford.

In §3 we give a sufficient condition (Theorem 2) for an arc of continua to satisfy the cellularity criterion. Applying this result to 1-complexes, we show (Theorem 3) that a contractible 1-complex K topologically embedded in the interior of an n-manifold M, $n \geqq 3$, satisfies the cellularity criterion if for each arc A in K, $M-A$ is 1-LC at an endpoint of A. Another result (Theorem 1) shows that a K' has this property provided each arc in K' is LPU at an endpoint. If

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the ambient manifold were E^n or S^n, a result of Doyle [8] could be used to conclude that each arc in such a K' satisfies the cellularity criterion; however, by the examples of §4, this would not in itself imply that K' satisfies the cellularity criterion.

2. **Definitions and notations.** We use E^n to denote n-dimensional Euclidean space with its usual metric, D^n to denote the unit ball in E^n, and S^n to denote the unit sphere in E^{n+1}. If M is a manifold, we use Bd M and Int M to denote the boundary and interior of M. Throughout, the term *manifold* is used in the topological sense; that is, we never assume that a manifold supports a piecewise linear or differentiable structure.

Let M be an n-manifold and $X \subset \text{Int } M$ a continuum. We say that X is UV^∞ if for each open set $U \supset X$ there exists an open set V such that $X \subset V \subset U$ and V is contractible in U. This is actually a topological property of X; that is, if $X \subset \text{Int } M$ is UV^∞, then any homeomorph of X in any manifold is also UV^∞ [2]. X is said to satisfy the *cellularity criterion* if for each open set $U \supset X$ there exists an open set V such that $X \subset V \subset U$ and each loop (map of S^1) in $V - X$ is contractible in $U - X$. McMillan showed [14, Theorems 1 and 1′] that if X is UV^∞, if $n \geqq 5$ and M supports a piecewise linear structure (or if $n = 3$ and $M = E^3$ or S^3), then X is cellular if and only if X satisfies the cellularity criterion. By [13, Theorem 3], the condition that M support a piecewise linear structure if $n \geqq 5$ can be removed. By [16, Corollary], the condition that M be E^3 or S^3 if $n = 3$ can be removed if X is 1-dimensional.

Suppose $B \subset A$ are closed nonempty subsets of M and that B contains limit points of $M - A$. Then $M - A$ is 1-LC at B if for each open set U containing B there exists an open set V containing B such that each loop in $V - A$ is contractible in $U - A$. If B consists of a single point, this coincides with the usual definition [9]. If $A \subset M$ is an arc and p is an endpoint of A, then A is LPU at p if there are arbitrarily small n-cells whose interiors contain p and whose boundaries intersect A in a single point [12]. We have stated these definitions only for the special cases required in this paper; their generalizations can be found in [9] and [12].

If A is an arc with endpoints p and q, we shall sometimes find it convenient to denote A by $[pq]$, $A - p$ by $(pq]$, etc. We use $[a, b]$ to denote the real numbers t such that $a \leqq t \leqq b$.

3. **Some conditions which imply the cellularity criterion.**

THEOREM 1. *Let A be an arc in the interior of the n-manifold M. If* *each subarc of A is* LPU *at an endpoint, then $M-A$ is* 1-LC *at an endpoint of A.*

PROOF. We denote A by $[pq]$ and make the supposition that for each $x\in(pq]$ the arc $[px]$ is LPU at x. For if this is not the case for some x, then for each $y\in[px)$ the arc $[yq]$ is LPU at y, and the proof below would show that $M-A$ is 1-LC at p.

Let U be an open set in M containing q. By our assumption, there exists an n-cell C such that $q\in C\subset U$ and $A\cap \text{Bd } C$ consists of a single point, say a. We claim that each loop in Int $C-A$ is contractible in $U-A$ (in fact, in Int $C-A$); hence $M-A$ is 1-LC at q.

Let $f:S^1\to \text{Int } C-A$ be a loop and let $B=\{x\in A \mid f$ is contractible in Int $C-[px]\}$. Since f is contractible in Int C we have $[pa]\subset B$; hence $B\neq\emptyset$. It is also apparent, since in a metric space a positive distance separates compact sets, that B is open. We now show that B is closed. Let $b\in(aq]$ be a limit point of B. By our assumption, there exists an n-cell D such that $b\in D\subset \text{Int } C$ and $[pb]\cap \text{Bd } D$ consists of a single point, say d. We may also suppose that $f(S^1)\cap D=\emptyset$. Let t be a point of $B\cap(db)$. Then there exists a map $F:D^2\to \text{Int } C-[pt]$ such that $F|S^1=f$. Let $r:D-d\to \text{Bd } D-d$ be a retraction and let $g:\text{Int } C-d\to \text{Int } C-d$ be defined by $g(x)=r(x)$ if $x\in D-d$, $g(x)=x$ otherwise. Then the map $g\circ F$ shows that f is contractible in Int $C-[pb]$. Hence $b\in B$ and B is closed; therefore $q\in B$ and f is contractible in $U-A$.

REMARKS. (i) Using [**10**, Example 1.1] it is easy to construct in E^3 an arc which is LPU at each endpoint but whose complement fails to be 1-LC at both endpoints.

(ii) Let A be an arc of the simple closed curve described by Bing in [**3**]. Then E^3-A is 1-LC at each endpoint of A but A fails to be LPU at both endpoints. Hence the converse of Theorem 1 is false.

Let X be a continuum. We say that h *represents X as an arc of subcontinua* if h is a map from X onto $[0, 1]$ such that $h^{-1}(t)$ is connected for each t.

THEOREM 2. *Let K be a* UV^∞ *continuum in the interior of the n-manifold M, $n\geqq 3$. Suppose there exists a representation h of K as an arc of subcontinua such that*

(1) *$h^{-1}(t)$ satisfies the cellularity criterion, and*

(2) *$M-h^{-1}([a, b])$ is* 1-LC *at $h^{-1}(a)$ or $h^{-1}(b)$, $0\leqq a<b\leqq 1$.*

Then K satisfies the cellularity criterion. Hence, if $n\geqq 5$ or if $n=3$ and a neighborhood of K embeds in E^3, K is cellular.

PROOF. We first consider the special case where $M-h^{-1}([a, b])$ is

1-LC at $h^{-1}(b)$ for $0 \leqq a < b \leqq 1$. Let U be an open subset of M and $K \subset U$. Since K is UV^∞, there is an open set V such that $K \subset V \subset U$ and V is contractible in U. We will show that K satisfies the cellularity criterion by showing that each loop in $V-K$ is contractible in $U-K$.

Let $f: S^1 \to V-K$ be a loop, and let $B = \{t \in [0, 1] \mid f$ is contractible in $U - h^{-1}([0, t])\}$. We first argue that $B \neq \emptyset$. By our choice of V, f extends to $f_*: D^2 \to U$. Now, since $h^{-1}(0)$ satisfies the cellularity criterion, there exists an open set V_0 such that $h^{-1}(0) \subset V_0 \subset U$ and each loop in $V_0 - h^{-1}(0)$ is contractible in $U - h^{-1}(0)$. Now, $C_0 = f_*^{-1}(h^{-1}(0))$ is a compact subset of $\operatorname{Int} D^2$. Let $D_1, D_2, \cdots, D_j$ be a covering of C_0 by pairwise disjoint disks such that $D_i \subset \operatorname{Int} D^2$ and $f_*(\operatorname{Bd} D_i) \subset V_0 - h^{-1}(0)$. By choice of V_0 there exist maps $f_i: D_i \to U - h^{-1}(0)$ such that $f_i \mid \operatorname{Bd} D_i = f_* \mid \operatorname{Bd} D_i$. Define $g: D^2 \to U$ by $g(x) = f_i(x)$ if $x \in D_i$, $g(x) = f_*(x)$ otherwise. Then g shows that $0 \in B$, hence $B \neq \emptyset$. The usual compactness argument shows that B is open.

To show that $1 \in B$ it only remains to show that B is closed. With this in mind, let $b > 0$ be a limit point of B. By our assumption, there exists an open set $V_b \subset U$ such that $h^{-1}(b) \subset V_b$ and each loop in $V_b - h^{-1}([0, b])$ is contractible in $U - h^{-1}([0, b])$. Since b is a limit point of B, there exists $t \in B \cap [0, b)$ such that $h^{-1}([t, b]) \subset V_b$. Since $t \in B$, there exists a map $F: D^2 \to U - h^{-1}([0, t])$ such that $F \mid S^1 = f$. Then $C = F^{-1}(h^{-1}([t, b]))$ is a compact subset of $\operatorname{Int} D^2$. Let $E_1, E_2, \cdots, E_k$ be a covering of C by pairwise disjoint disks such that $E_i \subset \operatorname{Int} D^2$ and $F(\operatorname{Bd} E_i) \subset V_b - h^{-1}([0, b])$. By our choice of V_b, there exist maps $g_i: E_i \to U - h^{-1}([0, b])$ such that $g_i \mid \operatorname{Bd} E_i = F \mid \operatorname{Bd} E_i$. Define $F_*: D^2 \to U$ by $F_*(x) = g_i(x)$ if $x \in E_i$, $F_*(x) = F(x)$ otherwise. Then F_* shows that $b \in B$, hence B is closed.

Now suppose the condition of the special case does not hold. Then there exists $t \in (0, 1]$ such that $M - h^{-1}([0, t])$ is not 1-LC at $h^{-1}(t)$. Let p be the least upper bound of the set of all such t's. By the "mirror-image" of the special case, $h^{-1}([0, p])$ satisfies the cellularity criterion. The proof now proceeds just as in the special case with $h^{-1}(0)$ being replaced by $h^{-1}([0, p])$.

Remarks. (iii) Let A denote the arc [**10**, Example 1.1]. Then there exists a noncellular disk $D \subset E^3$ such that A spans D and D is locally tame modulo A. It is easy to see that there is a representation h_1 of D as an arc of subcontinua such that $h_1^{-1}(0)$ and $h_1^{-1}(1)$ are the endpoints of A while $h_1^{-1}(t)$ is a tame arc if $0 < t < 1$. Hence condition (2) cannot be eliminated from the hypothesis of Theorem 2. We also note that in this example $E^3 - h_1^{-1}([a, b])$ is 1-LC at $h_1^{-1}(a)$ or $h_1^{-1}(b)$ unless $a = 0$ and $b = 1$. We can also represent D as an arc of subcontinua by h_2 where $h_2^{-1}(1/2)=A$ and $h_2^{-1}(t)$ is a tame arc if $t\neq 1/2$. $E^3-h_2^{-1}([a, b])$ is 1-LC at $h_2^{-1}(a)$ (or $h_2^{-1}(b)$) if $a\neq 1/2$ (or $b\neq 1/2$). Hence condition (1) is also necessary for the proof of Theorem 2.

(iv) The idea of the proof of Theorem 2 can also be used to obtain some results on unions of continua. For example: *Suppose C_1, C_2, and $C_1\cup C_2$ are* UV$^\infty$ *continua in the interior of the n-manifold M, $n\geqq 3$. Then $C_1\cup C_2$ satisfies the cellularity criterion if C_1 satisfies the cellularity criterion and $M-(C_1\cup C_2)$ is* 1-LC *at C_2.*

THEOREM 3. *Let K be a contractible* 1*-complex topologically embedded in the interior of the n-manifold M, $n\geqq 3$. If $M-A$ is* 1-LC *at an endpoint of A for each arc $A\subset K$, then K satisfies the cellularity criterion. Hence, if $n\neq 4$, K is cellular.*

PROOF. Each contractible 1-complex K has an abstract simplicial triangulation $T(K)$ which is "minimal" in the sense that no vertex of $T(K)$ is a face of exactly two 1-simplexes. We prove the theorem by induction on the number of 1-simplexes, $\#(K)$, of $T(K)$. If $\#(K)=1$, then K is an arc and the result follows immediately from Theorem 2. If K is not an arc, let $[ab]$ and $[cd]$ be 1-simplexes of $T(K)$ such that a (resp. d) is a vertex of no 1-simplex other than $[ab]$ (resp. $[cd]$). We suppose for the moment that there exists a subarc $[xb]$ of $[ab]$ such that $M-[xb]$ fails to be 1-LC at x. Then for each subarc $[cy]$ of $[cd]$, $M-[cy]$ is 1-LC at y. For given such a y, there is an arc in K from x to y. Let L denote $K-(cd]$. Then $\#(L)=\#(K)-1$, and we may assume inductively that L satisfies the cellularity criterion. Now let $h:K\to[0, 1]$ be a map such that $h(L)=0$ and $h|(c, d]:(c, d]\to(0, 1]$ is a homeomorphism. Since h satisfies conditions (1) and (2) of Theorem 2, K satisfies the cellularity criterion. Now, if for each subarc $[xb]$ of $[ab]$, $M-[xb]$ is 1-LC at x, we let $L=K-[ab)$ and proceed as above.

REMARK. (v) If $[ab]$ denotes a maximal arc of the triod K_3 constructed in the next section, then $E^3-[ab]$ fails to be 1-LC at a and at b. But $[ab]$ is cellular. Thus the converse of Theorem 3 is false.

Now, Theorems 1 and 2 immediately imply:

COROLLARY 4. *Let K be a contractible* 1*-complex topologically embedded in the interior of an n-manifold, $n\geqq 3$. If A is* LPU *at an endpoint of A for each arc $A\subset K$, then K satisfies the cellularity criterion. Hence, if $n\neq 4$. K is cellular.*

REMARK. (vi) If the ambient manifold in Corollary 4 were E^n or S^n, the result would follow from a theorem of Stewart [**18**, Theorem 1];

one should note that [18, Theorem 1] is false as stated, but can be modified appropriately.

4. A condition which does not imply the cellularity criterion. By an *n-frame* we mean the union of n arcs intersecting only in a common endpoint. Debrunner and Fox [7] have constructed, for $n \geqq 3$, an n-frame in E^3 which is wild but each of whose $(n-1)$-frames is tame. In this section we construct, for $n \geqq 3$, an n-frame K_n in E^3 which is noncellular but each of whose $(n-1)$-frames is cellular. By construction, K_n shall lie on a sphere which is locally tame modulo K_n. The construction is quite similar to ones given in [1], [4], and [11]. Our brief description shall assume familiarity with these references, particularly [1].

If $i=1, 2, \cdots$, or n and $j=1, 2, \cdots$, or 13, let E_{ij} be the disk bounded by the union of the sets given, in polar coordinates, by $\{(r, \theta) \mid r=13-j \text{ or } 14-j, \ (4\pi i-\pi)/2n \leqq \theta \leqq (4\pi i+\pi)/2n\}$ and $\{(r, \theta) \mid 13-j \leqq r \leqq 14-j, \theta=(4\pi i-\pi)/2n \text{ or } (4\pi i+\pi)/2n\}$. As in [1], we thicken each E_{ij} slightly and add to each thickened E_{ij} a solid feeler with solid torus H_{ij}. The loop of H_{i1} circles the stem of H_{i2}, the loop of H_{i2} circles the stem of $H_{i3}, \cdots$, the loop of $H_{i,12}$ circles the stem of $H_{i,13}$. See [1, Figure 2]. In addition, the loop of $H_{1,13}$ circles the stem of $H_{2,13}$, the loop of $H_{2,13}$ circles the stem of $H_{3,13}, \cdots$, the loop of $H_{n-1,13}$ circles the stem of $H_{n,13}$, and the loop of $H_{n,13}$ circles the stem of $H_{1,13}$. The loop of H_{ij} does not circle the stem of H_{rs} unless mentioned above.

Next, a slice is removed from each H_{ij} so that (sliced $H_{ij}) \cup$(thickened E_{ij}) is a 3-cell C_{ij}. Then E_{ij} separates Bd C_{ij} into two disks, one of which contains points of the sliced H_{ij}. The interior of this disk is pushed slightly into Int C_{ij} to form a disk D_{ij}. Then D_{ij} is divided into disks $E_{ij,1}, E_{ij,2}, \cdots, E_{ij,13}, A_{ij,1}, A_{ij,2}$ as in [1, Figure 3]. The intersection of $E_{i,13,13}$ and $E_{j,13,13}$ is the origin if $i \neq j$. We thicken the E_{ijk} and add solid feelers with solid tori H_{ijk}. For a fixed i, the linking of the H_{ijk}'s is as described in [1]. In addition, the loop of $H_{1,13,13}$ circles the stem of $H_{2,13,13}$, the loop of $H_{2,13,13}$ circles the stem of $H_{3,13,13}, \cdots$, the loop of $H_{n-1,13,13}$ circles the stem of $H_{n,13,13}$, and the loop of $H_{n,13,13}$ circles the stem of $H_{1,13,13}$. The loop of H_{ijk} does not circle the stem of H_{rst} unless mentioned above.

Continuing the process in the obvious way, we obtain K_n. It is the union of arcs $A_1, A_2, \cdots, A_n$ where $A_i \cap A_j$ is the origin if $i \neq j$. Using the proof of [4, Claim, p. 6] we find that $\pi_1(E^3-K_n) \neq 0$. Thus K_n is not cellular. For $k=1, 2, \cdots$, or n, let $J_k=(K_n-A_k) \cup$(origin). Then J_k is cellular. To see this, one can check to see that J_k satisfies the cellularity criterion; or one can construct in any open set containing J_k a 3-cell containing J_k. Details are left to the reader.

REMARK. (vii) Cantrell [6] has shown that there does not exist a "mildly wild" n-frame in E^q if $q \geqq 4$; that is, an n-frame in E^q is tame if each of its $(n-1)$-frames is. It would be interesting to determine if there is a noncellular n-frame in E^q, $q \geqq 4$, each of whose $(n-1)$-frames is cellular. Brown [5] has shown that there is a noncellular arc in E^q, $q \geqq 3$, each of whose proper subarcs is cellular.

ADDED IN PROOF. Using some recent results of J. L. Bryant, C. L. Seebeck III, and R. J. Daverman and W. T. Eaton, an answer can now be given to the question raised in Remark (vii). Specifically, one can construct, for each $n \geqq 3$ and $q \geqq 3$, an n-frame in E^q which is noncellular but each of whose $(n-1)$-frames is cellular. Details of the construction will appear in another paper.

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UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30601

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540.