SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

THE SOLUTION TO ONE OF ULAM'S PROBLEMS CONCERNING ANALYTIC SETS. II

R. MANSFIELD

ABSTRACT. It is shown that the universal analytic set is not a σ-combination of measurable rectangles.

This note is an extreme simplification of a previous paper bearing the same name [2]. The use of the Lebesgue measurable sets in the statement of the lemma is due to B. V. Rao [3]; it was while reading Rao's proof that I discovered the simplification given here.

Ulam asks whether there is an analytic subset of the unit square which is not in the σ-algebra generated by the analytic rectangles [4, p. 9]. We answer that question affirmatively. In fact we show that the universal analytic set is not in the σ-algebra generated by the measurable rectangles.

LEMMA. *Any countably generated σ-algebra consisting entirely of Lebesgue measurable sets does not contain all analytic sets.*

PROOF. Suppose the algebra $\mathfrak{A}$ is generated by the entries of the countable sequence of measurable sets $\langle A_n\rangle_{n\in\omega}$. We claim that there is an uncountable closed set P such that for every n, $A_n\cap P$ is closed. Then every set in $\mathfrak{A}$ would have a Borel intersection with P, but any uncountable closed set has an analytic, non-Borel subset [1, §34]. In order to prove our claim let us define a function F by induction. The domain of F is the set of all finite sequences of 0's and 1's. $\langle\cdot\rangle$ is the empty sequence. Choose $F(\langle\cdot\rangle)$ to be a compact set of positive Lebesgue measure which is either disjoint from or a subset of A_0. Given $F(s)$ for s a sequence of length n, choose $F(s*\langle 0\rangle)$ and $F(s*\langle 1\rangle)$ to be two disjoint compact subsets of $F(s)$, each of positive Lebesgue measure and each either a subset of or disjoint from A_{n+1}. Given a function α from the integers into the set $\{0, 1\}$, let us denote the sequence $\langle\alpha(0), \alpha(1), \cdots, \alpha(n)\rangle$ by the symbol $\bar{\alpha}(n)$. Then define $P=\bigcup_\alpha\bigcap_n F(\bar{\alpha}(n))$. For each α, $\bigcap_n F(\bar{\alpha}(n))$ is a decreasing intersection

Received by the editors September 18, 1969 and, in revised form, March 22, 1970.
AMS 1968 *subject classifications*. Primary 0440, 2810.
Key words and phrases. Analytic, Borel, measurable rectangle.

of compact sets and so is nonempty. Clearly $\alpha \neq \beta$ implies that $\bigcap_n F(\bar{\alpha}(n))$ is disjoint from $\bigcap_n F(\bar{\beta}(n))$. Thus P is uncountable. Furthermore P can also be written as $\bigcap_n \bigcup \{F(s)\colon \text{length } (s) = n\}$. Since there are only 2^n sequences of length n, this shows that P is an intersection of finite unions of compact sets, and so is itself compact. The intersection of P with A_n is equal to

$$P \cap \bigcup \{F(s)\colon \text{length } (s) = n \wedge A_n \subseteq F(s)\},$$

which is closed; completing the proof of the lemma.

Now let Σ be any analytic subset of the plane which is universal for analytic subsets of the line; i.e. for any linear analytic set A there is a number x such that $A = \{y\colon \langle x, y\rangle \in \Sigma\}$ [1, §34].

THEOREM. *Σ is not in the σ-algebra generated by the measurable rectangles.*

PROOF. Suppose otherwise. Then Σ is in a σ-algebra generated by a countable set of measurable rectangles. So suppose Σ is in the σ-algebra generated by the entries of the sequence $\langle A_n \times B_n \rangle_{n \in \omega}$. But for A an arbitrary linear analytic set there is an x with $A = \{y\colon \langle x, y\rangle \in \Sigma\}$. Thus A is in the σ-algebra generated by the sequence $\langle B_n^* \rangle$, where $B_n^* = B_n$ if $x \in A_n$ and $B_n^* = 0$ if $x \notin A_n$. Thus every linear analytic set is in the σ-algebra generated by $\{B_n\colon n \in \omega\} \cup \{0\}$.

BIBLIOGRAPHY

1. L. Kuratowski, *Topology*. Vol. 1, PWN, Warsaw and Academic Press, New York, 1966, MR **36** #840.

2. R. Mansfield, *The solution to one of Ulam's problems concerning analytic rectangles*, Proc. Summer Institute on Set Theory, U.C.L.A., Los Angeles, Calif., 1967.

3. B. V. Rao, *A remark on analytic sets*, Indian Statistical Institute, Calcutta, 1968.

4. S. M. Ulam, *Problems in modern mathematics*, Wiley, New York, 1964.

UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024