LINEAR DIFFERENTIAL-DIFFERENCE OPERATORS AND THEIR ADJOINTS[1]

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ABSTRACT. The formal adjoint for a first order matrix differential-difference operator is shown to be a true Hilbert space adjoint, and conditions under which such operators are selfadjoint (in a Hilbert space sense) are derived. Differential-difference operators whose domains are defined by a given initial function cannot be selfadjoint, whereas certain operators whose domains are defined only by conditions at the endpoints of an interval can be selfadjoint.

In the past several years there has been considerable interest in differential-difference equations (hereafter abbreviated by D. D. equations) or differential equations with retarded argument. A large majority of the publications in this area have dealt with existence theory or stability theory for such equations. There seem to have been very few results in the direction of boundary value problems and adjoints for a D. D. equation. A. Halanay has given some results concerned with boundary value problems in [4]. Also both he and R. Bellman and K. Cooke use the formal adjoint of a D. D. equation to obtain an analog of the variations of parameter formula [3, pp. 359–362] and [2, pp. 320–323].

This article considers the problem of determining an adjoint of a D. D. equation from the point of view of linear operators in Hilbert space. We define two distinct operators, and in particular we investigate the possibility of their being selfadjoint as operators on a dense subset of a Hilbert space. As far as is known to the author, this is the first paper in which it is shown that the formal adjoint is also a Hilbert space adjoint.

Equations like those which we consider arise as Euler-type equations for variational problems with delayed argument having quadratic integrand. See for example Hughes [5].

Throughout this paper A_i, B_i denote continuous n by n matrices defined on $[a-\tau, b+\tau]$ where τ represents a positive real number.

We also suppose that M and N are m by n constant matrices such that the matrix $(M:N)$ is of rank m, $m \leq 2n$. A prime will always denote differentiation.

Problem I. Let H_0 be the Hilbert space of n-dimensional vector functions x defined on $[a-\tau, b+\tau]$ whose components are in L^2 and $x(t) = 0$, $a-\tau \leq t < a$. The inner product is given by $\langle x, y \rangle = \int_a^{b+\tau} y^* x$ where y^* is the transpose of y.

Define the boundary operator $\mu(x)$ and the differential-difference expression $E(x)$ by the following formulas:

$$\mu(x) = Mx(a) + Nx(b + \tau),$$

$$E(x) = A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t - \tau) + A_2(t)\dot{x}(t + \tau)$$
$$+ B_0(t)x(t) + B_1(t)x(t - \tau) + B_2(t)x(t + \tau), \quad a \leq t \leq b,$$

$$E(x) = A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t - \tau) + B_0(t)x(t) + B_1(t)x(t - \tau),$$
$$b \leq t \leq b + \tau,$$

where A_1, A_2, B_1, B_2 are not all zero.

We denote by D the set of all x in H_0 with the following properties:
(i) x is absolutely continuous on $[a, b+\tau]$.
(ii) $E(x)$ is in H_0.

Let us now define a differential-difference operator $L: D \to H_0$ by setting $Lx = E(x)$ for all x in D.

Since D is dense in H_0, there is a well-defined adjoint operator L^*.

THEOREM 1. *If z is in the domain of L^*, then*

$$L^*z = -[A_0^*(t)z(t) + A_1^*(t+\tau)z(t+\tau)]' + [B_0^*(t)z(t) + B_1^*(t+\tau)z(t+\tau)],$$
$$a \leq t < a + \tau;$$

$$L^*z = -[A_0^*(t)z(t) + A_1^*(t+\tau)z(t+\tau) + A_2^*(t-\tau)z(t-\tau)]'$$
$$+ [B_0^*(t)z(t) + B_1^*(t+\tau)z(t+\tau) + B_2^*(t-\tau)z(t-\tau)], \quad a + \tau \leq t < b;$$

$$L^*z = -[A_0^*(t)z(t) + A_2^*(t-\tau)z(t-\tau)]' + [B_0^*(t)z(t) + B_2^*(t-\tau)z(t-\tau)],$$
$$b \leq t \leq b + \tau.$$

Let D_0 be the set of all x in D such that

$$x(a) = 0 = x(b) = x(b+\tau) = x(a+\tau).$$

Let x be in D_0 and solve the equation

$$\langle Lx, z \rangle = \langle x, \theta \rangle \quad \text{for } \theta.$$

$$0 = \int_a^b z^*(t)[A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t - \tau) + A_2(t)\dot{x}(t + \tau)$$

$$+ B_0(t)x(t) + B_1(t)x(t - \tau) + B_2(t)x(t + \tau)]dt$$

$$- \int_a^{b+\tau} \theta^* x(t)dt$$

$$+ \int_b^{b+\tau} z^*(t)[A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t - \tau)$$

$$+ B_0(t)x(t) + B_1(t)x(t - \tau)]dt.$$

In the terms involving $x(t-\tau)$ and $\dot{x}(t-\tau)$ let $s = t - \tau$. Make the substitution $s = t + \tau$ in the terms with $x(t+\tau)$ and $\dot{x}(t+\tau)$. The preceding expression then becomes

$$0 = \int_a^{a+\tau} \{[z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau)]\dot{x}(t)$$

$$+ [z^*(t)B_0(t) + z^*(t + \tau)B_1(t + \tau) - \theta^*]x(t)\}dt$$

$$+ \int_{a+\tau}^b \{[z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau) + z^*(t - \tau)A_2(t - \tau)]\dot{x}(t)$$

$$+ [z^*(t)B_0(t) + z^*(t + \tau)B_1(t + \tau)$$

$$+ z^*(t - \tau)B_2(t - \tau) - \theta^*]x(t)\}dt$$

$$+ \int_b^{b+\tau} \{[z^*(t)A_0(t) + z^*(t - \tau)A_2(t - \tau)]\dot{x}(t)$$

$$+ [z^*(t)B_0(t) + z^*(t - \tau)B_2(t - \tau) - \theta^*]x(t)\}dt.$$

Integrating by parts and noting that x is in D_0, we have the following equation: $0 = \int_a^{b+\tau} \psi(t)\dot{x}(t)dt$ where

$$\psi(t) = z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau)$$

$$- \int_a^t [z^*(s)B_0(s) + z^*(s + \tau)B_1(s + \tau) - \theta^*]ds, \quad a \leqq t < a + \tau,$$

$$\psi(t) = z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau) + z^*(t - \tau)A_2(t - \tau)$$

$$- \int_{a+\tau}^t [z^*(s)B_0(s) + z^*(s + \tau)B_1(s + \tau)$$

$$+ z^*(s - \tau)B_2(s - \tau) - \theta^*]ds, \quad a + \tau \leqq t < b,$$

$$\psi(t) = z^*(t)A_0(t) + z^*(t-\tau)A_2(t-\tau)$$

$$- \int_b^t [z^*(s)B_0(s) + z^*(s-\tau)B_2(s-\tau) - \theta^*]ds, \quad b \leq t \leq b+\tau.$$

Now the fundamental lemma of the calculus of variations yields that ψ is constant on each of the subintervals $(a, a+\tau)$, $(a+\tau, b)$, and $(b, b+\tau)$. Upon differentiation of ψ and solution for θ, we arrive at the result of the theorem.

The result of Theorem 1 shows that the operator L defined by $Lx = E(x)$ on D cannot be selfadjoint. However, using this result as a guide, we now consider a D. D. operator whose adjoint has the same form as the original operator.

Problem II. Let H be the Hilbert space of all n-dimensional vector functions defined on $[a-\tau, b+\tau]$ whose components are square integrable. The inner product is given by $\langle x, y \rangle = \int_{a-\tau}^{b+\tau} y{*}x$.

Define the boundary operator λ by the equation

$$\lambda(x) = Mx(a - \tau) + Nx(b + \tau)$$

and define the symbol $D(x)$ by the following D. D. expressions:

$$D(x) = A_0(t)\dot{x}(t) + A_2(t)\dot{x}(t+\tau) + B_0(t)x(t) + B_2(t)x(t+\tau), \quad a-\tau \leq t < a;$$

$$D(x) = A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t-\tau) + A_2(t)\dot{x}(t+\tau)$$

$$+ B_0(t)x(t) + B_1(t)x(t-\tau) + B_2(t)x(t+\tau), \quad a \leq t < b;$$

$$D(x) = A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t-\tau) + B_0(t)x(t) + B_1(t)x(t-\tau), \quad b \leq t \leq b+\tau.$$

Let B be the set of all x in H which satisfy the conditions:
(i) x is absolutely continuous on $[a-\tau, b+\tau]$.
(ii) $D(x)$ is in H.

We now define an operator A on B to H by the formula $Ax = D(x)$ for x in B. Since B is dense in H, there exists a well-defined adjoint operator A^*. Let B^* be the domain of A^*.

THEOREM 2. *If z is in B^*, then*

$$A^*z = -[A_0^*(t)z(t) + A_1^*(t+\tau)z(t+\tau)]' + B_0^*(t)z(t) + B_1^*(t+\tau)z(t+\tau),$$

$$a-\tau \leq t \leq a,$$

$$A^*z = -[A_0^*(t)z(t) + A_1^*(t+\tau)z(t+\tau) + A_2^*(t-\tau)z(t-\tau)]$$

$$+ B_0^*(t)z(t) + B_1^*(t+\tau)z(t+\tau) + B_2^*(t-\tau)z(t-\tau), \quad a \leq t \leq b,$$

$$A^*z = -[A_0^*(t)z(t) + A_2^*(t-\tau)z(t-\tau)]' + B_0^*(t)z(t) + B_2^*(t-\tau)z(t-\tau),$$

$$b \leq t \leq b+\tau.$$

Again we solve the equation $0 = \langle Ax, z\rangle - \langle x, \theta\rangle$ for θ. We must again restrict x to the set $B_0 = \{x \text{ in } B : x(a-\tau) = x(a) = x(b) = x(b+\tau) = 0\}$. The calculations are quite analogous to those used in the proof of Theorem 1.

Let B^0 be the set of all x in B with the property that $\lambda(x) = 0$. Let B^0 be the domain of A and let B^{0*} be the domain of $A*$.

THEOREM 3. *If z is in B^{0*}, then z satisfies the following boundary conditions*:

$$A_0^*(a - \tau)z(a - \tau) + A_1^*(a)z(a) = -M^*\sigma,$$
$$A_0^*(b + \tau)z(b + \tau) + A_2^*(b)z(b) = N^*\sigma$$

where σ is an m-dimensional vector.

Let x be in B and z in B^*. Then

$\langle Ax, z\rangle - \langle x, A^*z\rangle$

$$= \int_{a-\tau}^{a} \{z^*(t)[A_0(t)\dot{x}(t) + A_2(t)\dot{x}(t + \tau) + B_0(t)x(t) + B_2(t)x(t + \tau)]$$

$$+ \{[z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau)]'$$
$$- z^*(t)B_0(t) - z^*(t + \tau)B_1(t + \tau)\}x(t)\}dt$$

$$+ \int_{a}^{b} \{z^*(t)[A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t - \tau) + A_2(t)\dot{x}(t + \tau)$$

$$+ B_0(t)x(t) + B_1(t)x(t - \tau) + B_2(t)x(t + \tau)]$$
$$+ \{[z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau) + z^*(t - \tau)A_2(t - \tau)]'$$
$$- z^*(t)B_0(t) - z^*(t + \tau)B_1(t + \tau) - z^*(t - \tau)B_2(t - \tau)\}x(t)\}dt$$

$$+ \int_{b}^{b+\tau} \{z^*(t)[A_0(t)\dot{x}(t) + A_1(t)\dot{x}(t - \tau) + B_0(t)x(t) + B_1(t)x(t - \tau)]$$

$$+ \{[z^*(t)A_0(t) + z^*(t - \tau)A_2(t - \tau)]'$$
$$- z^*(t)B_0(t) - z^*(t + \tau)B_1(t + \tau)\}x(t)\}dt$$

$$= \int_{a-\tau}^{a} \{[z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau)]x(t)\}'dt$$

$$+ \int_{a}^{b} \{[z^*(t)A_0(t) + z^*(t + \tau)A_1(t + \tau)$$

$$+ z^*(t - \tau)A_2(t - \tau)]x(t)\}'dt$$

$$+ \int_{b}^{b+\tau} \{[z^*(t)A_0(t) + z^*(t - \tau)A_2(t - \tau)]x(t)\}'dt.$$

Now we have the equations

$$
\begin{aligned}
0 = {}& [z^*(a-)A_0(a) + z^*(a+\tau-)A_1(a+\tau)]x(a) \\
& - [z^*(a-\tau)A_0(a-\tau) + z^*(a+)A_1(a)]x(a-\tau) \\
& + [z^*(b-)A_0(b) + z^*(b+\tau)A_1(b+\tau) + z^*(b-\tau-)A_2(b-\tau)]x(b) \\
& - [z^*(a+)A_0(a) + z^*(a+\tau+)A_1(a+\tau) + z^*(a-\tau)A_2(a-\tau)]x(a) \\
& + [z^*(b+\tau)A_0(b+\tau) + z^*(b-)A_2(b)]x(b+\tau) \\
& - [z^*(b+)A_0(b) + z^*(b-\tau+)A_2(b-\tau)]x(b) \\
= {}& [z^*(a-)A_0(a) + z^*(a+\tau-)A_1(a+\tau) - z^*(a+)A_0(a) \\
& \qquad - z^*(a+\tau+)A_1(a+\tau) - z^*(a-\tau)A_2(a-\tau)]x(a) \\
& + [z^*(b-)A_0(b) + z^*(b+\tau)A_1(b+\tau) + z^*(b-\tau-)A_2(b-\tau) \\
& \qquad - z^*(b+)A_0(b) - z^*(b-\tau+)A_2(b-\tau)]x(b) \\
& + [z^*(b+\tau)A_0(b+\tau) + z^*(b-)A_2(b)]x(b+\tau) \\
& - [z^*(a-\tau)A_0(a-\tau) + z^*(a+)A_1(a)]x(a-\tau).
\end{aligned}
$$

We conclude that each of the first two expressions in brackets is zero as well as

$$
\begin{aligned}
& [z^*(b+\tau)A_0(b+\tau) + z^*(b-)A_2(b)]x(b+\tau) \\
& \quad - [z^*(a-\tau)A_0(a-\tau) + z^*(a+)A_1(a)]x(a-\tau) = 0.
\end{aligned}
$$

Hence there exists a constant m-dimensional vector σ such that

$$
z^*(a-\tau)A_0(a-\tau) + z^*(a+)A_1(a) = -\sigma^* M
$$

and

$$
z^*(b+\tau)A_0(b+\tau) + z^*(b-)A_2(b) = \sigma^* N.
$$

The conclusion of the theorem follows by transposition.

It is to be noted that as a result of the proof of Theorem 3, we find that the differentiated portion of the operator A^* is absolutely continuous on $[a-\tau, b+\tau]$.

Also as a result of Theorem 3, we find that A cannot be selfadjoint. However if $A_1 = A_2 = 0$ on $[a-\tau, b+\tau]$, then A has the possibility of being selfadjoint. The following theorem is proved as in Reid [6].

THEOREM 4. *The operator A with $A_1 = A_2 = 0$ is selfadjoint if and only if $A_0 = -A_0^*$, $B_0 = B_0^* - A_0^{*\prime}$, $B_1^*(t+\tau) = B_2(t)$, and $M A_0^*(a-\tau)^{-1}M^* = N A_0^*(b+\tau)^{-1}N^*$.*

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