A ROOT OF UNITY OCCURRING IN PARTITION THEORY

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ABSTRACT. In this paper a new representation is found for the root of unity occurring in the well-known transformation equation of the generating function for $p(n)$, the number of partitions of the positive integer n.

Let $p(n)$ denote the number of partitions of the positive integer n. In the transformation equation for the generating function of $p(n)$ (see [1] and [3]) there occurs a certain root of unity which we shall denote by $\omega(h, k)$. Here k is a positive integer and h is an integer coprime to k. $\omega(h, k)$ also appears in the exponential sums $A_k(n)$ which occur in the infinite series represention of $p(n)$ due to Rademacher [3]. It should be mentioned, however, that a formula for $A_k(n)$ has been found by Selberg which does not depend on $\omega(h, k)$ (see [5] and [6]).

In [1] it was shown by Hardy and Ramanujan that

$$\omega(h, k) = (-h \mid k) \exp\{-\pi i((k-1)/4 + (k^2-1)(2h + H - h^2H)/12k)\}$$

if k is odd, and

$$\omega(h, k) = (-k \mid h) \exp\{-\pi i((2 - hk - h)/4 + (k^2-1)(2h + H - h^2H)/12k)\}$$

if k is even. Here, and in the sequel, $(a \mid b)$ is the Jacobi symbol while H is any solution of the congruence $hH \equiv 1 \pmod{k}$. In [2] Rademacher showed that $\omega(h, k) = \exp\{\pi i s(h, k)\}$ where $s(h, k)$ is a Dedekind sum defined by $s(h, k) = \sum_{u=1}^{k} ((u/k))((hu/k))$ with $((x)) = 0$ if x is an integer and $((x)) = x - [x] - 1/2$ otherwise.

The purpose of the present note is to present still another representation of $\omega(h, k)$ which appears to be somewhat simpler to handle in computations than those just stated. Thus, we shall prove the following

THEOREM. *If k is odd then*

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$$\omega(h, k) = (h \mid k) i^{(k-1)/2} \exp\{2\pi i q(h - h')/gk\}; \tag{1}$$

if k is even then

$$\omega(h, k) = (k \mid h) i^{b(k+1)/2} \exp\{2\pi i q(h - h')/gk\}. \tag{2}$$

If $J = (k, 3)$ then $g = J$ or $8J$ according as k is odd or even. h' is any solution of $hh' \equiv -1 \pmod{gk}$, and q is any solution of $24q/g = 1 \pmod{gk}$. In (2) $h' \equiv b \pmod 8$, and the branch of $i^{b(k+1)/2}$ is that corresponding to the principal value of $\log z$.

Our proof is based on four lemmas concerning the Dedekind sums $s(h, k)$. These are essentially Theorems 17, 18, 19 in [4].

LEMMA 1. $12ks(h, k) \equiv 0 \pmod 3$ *if and only if* $J = 1$.

LEMMA 2. $12ks(h, k) \equiv h - h' \pmod{Jk}$.

LEMMA 3. *If k is odd, then* $12ks(h, k) \equiv k + 1 - 2(h \mid k) \pmod 8$.

LEMMA 4. *If $k = 2^a K$, $a > 0$ and K odd, then*

$$12ks(h, k) \equiv h - h' - h'k^2 - 3h'k + 6h'k(k \mid h) \pmod{2^{a+3}}.$$

Now suppose first that k is odd and let $f = 24/J$. Then

$$12ks(h, k) \equiv 9k + 9 + 6(h \mid k) \pmod f. \tag{3}$$

For if $f = 8$ then (3) follows immediately from Lemma 3, while if $f = 24$ then (3) follows from Lemmas 1, 3 and the Chinese Remainder Theorem.

If we define the integers q and r by the congruences

$$fq \equiv 1 \pmod{Jk}, \qquad kr \equiv 1 \pmod f, \tag{4}$$

then it follows from Lemma 2, (3), (4) that

$$12ks(h, k) \equiv kr(9k + 9 + 6(h \mid k)) + fq(h - h') \pmod{24k}.$$

Therefore,

$$\begin{aligned} \omega(h, k) &= \exp\{2\pi i(12ks(h, k)/24k)\} \\ &= \exp\{2\pi i(r(9k + 9 + 6(h \mid k))/24 + q(h - h')/Jk)\}. \end{aligned}$$

Since $9 = -9 + 18$ and since $2r(18 + 6(h \mid k)/24$ is even or odd according as $(h \mid k) = 1$ or $(h \mid k) = -1$, respectively, we see that

$$\omega(h, k) = (h \mid k) \exp\{2\pi i q(h - h')/Jk\} \exp\{3\pi i r(k - 1)/4\}.$$

From (4) we have $r \equiv k \pmod 8$, and since $\exp\{3\pi i k(k-1)/4\} = i^{(k-1)/2}$ the proof of (1) is complete.

If k is even let $f = 24/8J$. Then

$$(5) \qquad 12ks(h, k) \equiv 0 \pmod{f}$$

is immediate if $f=1$ and follows from Lemma 1 if $f=3$. From Lemmas 2 and 4 we have

$$(6) \qquad 12ks(h, k) \equiv h - h'(1 + k^2 + 3k - 6k(k \mid h)) \pmod{8Jk},$$

and if q is defined by the congruence

$$(7) \qquad fq \equiv 1 \pmod{8Jk}$$

we see from (5) and (6) that

$$12ks(h, k) \equiv fq(h - h'(1 + k^2 + 3k - 6k(k \mid h))) \pmod{24k}.$$

Therefore,

$$\begin{aligned} \omega(h, k) &= \exp\{2\pi i(12ks(h, k)/24k)\} \\ &= \exp\{2\pi iq(h - h'(1 + k^2 + 3k - 6k(k \mid h)))/8Jk\}. \end{aligned}$$

Since $3k \equiv -3k-18k \pmod{8Jk}$, and since $2qh'(18k+6k(k|h))/8Jk$ is even or odd according as $(k|h)=1$ or $(k|h)=-1$, we see that

$$(8) \quad \omega(h, k) = (k \mid h) \exp\{2\Pi iqh'(3 - k)/8J\} \exp\{2\Pi iq(h - h')/8Jk\}.$$

If $J=1$ then $f=3$, and from (7) we have $q\equiv 3 \pmod 8$ so that $qh'(3-k)\equiv h'(9-3k)\equiv h'(1+k) \pmod 8$. If $J=3$ then $f=1$, and from (7) we have $q\equiv 9 \pmod 8$ so that $qh'(1-k/3)\equiv h'(9-3k) \equiv h'(1+k) \pmod 8$. Since $\exp\{2\pi ih'(1+k)/8\} = \exp\{\pi ib(1+k)/4\}$ if $h'\equiv b \pmod 8$ we see that (2) follows from (8) and the proof of the theorem is complete.

We remark in closing that although $\omega(h, k)$ is almost always referred to in the literature as a $24k$th root of unity it is obvious from our theorem that it is also a $12k$th root of unity.

Added in proof. Since this paper was accepted for publication it has been brought to the author's attention that Professor T. M. Apostol observed that $\omega(h, k)$ is a $12k$th root of unity in his doctoral dissertation (University of California at Berkeley, 1948). His observation, however, was not published.

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