

COMPLETE IDEALS AND MONOIDAL TRANSFORMS

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ABSTRACT. It is proved that the monoidal transform of an integral noetherian scheme with respect to a sheaf I of ideals is normal if and only if high powers of I are complete. The analogous theorem for linear systems is included, and as an application, it is proved that a rational singularity is absolutely isolated.

A key geometric result of the elementary theory of linear systems is that a projective variety is normal if and only if the hypersurface sections of high degree form a complete linear system. This theorem has an exact analogue in the theory of complete ideals: the monoidal transform of a noetherian domain with respect to an ideal I is normal if and only if high powers of I are complete. Though this result is implicitly contained in the appendix to Zariski-Samuel which works out the two parallel theories, as well as in papers of Muhly, Nagata, and especially Rees [3], it does not seem to be explicitly stated and proved anywhere. We do so here, treating the two theories together; the proof seems new even in the linear systems case. The only technical tool needed is the notion of superficial element, though projective cohomology could be used instead.

As an application, we prove that the first neighborhood of an isolated rational singularity of a surface is always normal.² This shows that such a singularity has only isolated singularities in its successive neighborhoods, never multiple curves; for a rational double point in characteristic zero, this was proved by Brieskorn.

1. Algebraic formulation. We recall briefly the essential parts of Zariski-Samuel [2]. Suppose one has a field K containing both a

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² ADDED IN PROOF. The long paper of Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, [Inst. Hautes Études Sci. Publ. Math. No. 36 (1969)] has just appeared. In it this theorem (among many others) is stated and proved. The first two sections of this paper heavily intersect Lipman's; he uses a different method to establish the completeness of m^n for a rational singularity.

domain A and an A -module I . The *integral closure* of I in K is

$$(1) \quad I' \equiv \{z \in K \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0, a_i \in I^i\}$$

and by an elementary theorem, this is also the *completion* of I in K :

$$(2) \quad I' \equiv \bigcap_{v \in \mathcal{S}} R_v I, \quad \mathcal{S} = \{v \mid R_v \supseteq A\},$$

where R_v is the valuation ring of the valuation v of K .

We introduce the graded algebras (we take $I^0 = A$)

$$(3) \quad R = \prod_{n \geq 0} I^n, \quad S = \prod_{n \geq 0} (I^n)'.$$

Note that (2) shows that $(I^m)'(I^n)' \subseteq (I^{m+n})'$, so that S is a graded R -algebra.

Let x_1, x_2, \dots be a set of nonzero generators for I , and consider the localizations

$$(4) \quad R_i = \left\{ \frac{F}{x_i^k} \mid F \in I^k \right\}, \quad S_i = \left\{ \frac{G}{x_i^k} \mid G \in (I^k)' \right\}.$$

Then R_i and S_i are subrings of K , and a routine argument using the integral equations (1) shows that

$$(5) \quad S_i = R_i'.$$

Since $x_{j_1} \cdots x_{j_n} = (x_{j_1}/x_i) \cdots (x_{j_n}/x_i)x_i^n$, one has in K

$$(6) \quad R_i I^n = R_i x_i^n, \quad S_i I^n = S_i x_i^n, \quad n \geq 0.$$

PROPOSITION 1. *If I is of finite type, $(I^n)' = \bigcap_i S_i I^n$, $n \geq 0$.*

PROOF. This is essentially [2, p. 354, Lemma]. Let \mathcal{S}_i be the set of valuations of K such that $R_v \supseteq R_i$. Then we claim $\mathcal{S} = \bigcup \mathcal{S}_i$, for if we are given any $v \in \mathcal{S}$, we find that index i for which $v(x_i)$ is minimum. Then $v(x_v/x_i) \geq 0$ for all v , so $v \in \mathcal{S}_i$. Thus

$$(I^n)' = \bigcap_i \bigcap_{v \in \mathcal{S}_i} R_v I^n = \bigcap_i \bigcap_{v \in \mathcal{S}_i} R_v x_i^n = \bigcap_i S_i x_i^n$$

according to (2), (6), and (5).

PROPOSITION 2. *If I is of finite type and A is noetherian,*

$$I^n = \bigcap_i R_i I^n \quad \text{for } n \geq n_0.$$

PROOF. This is really projective cohomology, but one can use superficial elements instead. Recall [3] that if \mathfrak{a} is an ideal in a noetherian

domain, $z \in \mathfrak{a}^d$ is *superficial of degree d for \mathfrak{a}* if

$$(7) \quad \mathfrak{a}^{n+d} : z = \mathfrak{a}^n \quad \text{for } n \geq n_0.$$

If z is such an element, then it follows immediately that

$$(8) \quad \mathfrak{a}^{n+dk} : z^k = \mathfrak{a}^n \quad \text{for } n \geq n_0, k > 0.$$

Superficial elements exist, though not necessarily for $d=1$.

The proof of the proposition takes place inside R . Let I be generated by x_1, \dots, x_m . Then in R we have the ideals

$$\bar{I} = \coprod_{n \geq 0} I^n, \quad \bar{I}^k = \coprod_{n \geq k} I^n.$$

R is noetherian, since it is finitely generated (by the x_i) over A , and it is also a graded subring of the graded ring $\bar{K} = \coprod \bar{K}_{(n)}$, where $\bar{K}_{(n)} = K$ for all $n \geq 0$. To prove the nontrivial inclusion \supseteq of the proposition, let $z \in \bar{I}^d$ be superficial of degree d for \bar{I} , and n_0 be as in (7). Suppose given $y \in K$ such that

$$y \in R_i I^n, \quad n \geq n_0, \text{ all } i.$$

We view y as an element of $\bar{K}_{(n)}$ in what follows. For some common exponent which we may take to be a multiple kd of the integer d ,

$$x_i y \in \bar{I}^{n+kd}, \quad n \geq n_0, \text{ all } i.$$

Since $z \in \bar{I}^d$, it follows in the usual way that if m is large

$$z^{km} y \in \bar{I}^{n+kd m}, \quad n \geq n_0.$$

Therefore $y \in \bar{I}^n$, according to (8). Thus $y \in \bar{I}^n \cap \bar{K}_{(n)} = I^n$. (If one is only interested in the case $I \subseteq A$, the same proof can be carried out entirely inside A .)

2. Geometric formulation. We translate the preceding section into geometry, using the simplest properties of the functor Proj on commutative graded algebras [4]. Let Y denote $\text{Spec } A$, and let V be $\text{Proj } R$ and W denote $\text{Proj } S$, so that we have the dual diagrams

$$(9) \quad \begin{array}{ccc} R & \longrightarrow & S \\ \nwarrow \pi^* & & \swarrow \pi \\ & A & Y \end{array} \quad \begin{array}{ccc} V & \longleftarrow & W \\ \searrow \pi & & \swarrow \nu \\ & Y & \end{array}$$

Since R is generated over A by the x_i , the definition of $\text{Proj } R$ shows that V is covered by the affines $V_i = \text{Spec } R_i$. It is easily checked that

$\pi^{-1}(V_i) = W_i = \text{Spec } S_i$. Then (5) shows that W_i is the normalization of V_i , so that without finiteness assumptions,

(10) $\text{Proj } S$ is the normalization in K of $\text{Proj } R$.

THEOREM 1. *If I^n is complete in K for large n , then $\text{Proj } R$ is normal. The converse holds, provided A is noetherian, I is of finite type, and K is the quotient field of $A[\dots, x_i/x_j, \dots]$.*

PROOF. If $I^n = (I^n)'$, then π^* is an isomorphism in high degree. An elementary property of Proj shows that π is then an isomorphism, so V is normal, by (10). Conversely, if V is normal and K is its function field, then by (10), $V = W$. Thus $V_i = W_i$ and $R_i = S_i$ for all i , so that $I^n = (I^n)'$ by Propositions 1 and 2, if $n \geq n_0$.

Let Y be an integral noetherian scheme, \mathcal{g} a coherent sheaf of fractional ideals on Y . The Y -scheme $V = \text{Proj}(\prod \mathcal{g}^n)$ is then called the monoidal transform of Y with respect to \mathcal{g} [4, 8.1.3]. To apply Theorem 1, we take K to be the function field of Y .

The completion \mathcal{g}' we define to be the sheaf such that $\Gamma(U, \mathcal{g}') = \Gamma(U, \mathcal{g})'$ for any affine $U \subset Y$. This defines a sheaf, as one sees easily using (2). If we now set $W = \text{Proj}(\prod (\mathcal{g}^n)')$, then W is the normalization of V . Routine globalization plus Proposition 1 gives

THEOREM 2. *The monoidal transform V of Y with respect to \mathcal{g} is a normal scheme if and only if \mathcal{g}^n is complete for n large. If W is the normalization of V , and $\nu: W \rightarrow Y$ the structural map,*

$$\nu_* \nu^* \mathcal{g}^n = (\mathcal{g}^n)' \quad \text{for } n \geq 0.$$

Consider the case of a linear system on a k -variety X , now. In Theorem 1, we take $Y = \text{Spec } k$ and $K = k(X)$. The module I is then a finite-dimensional k -space of functions on X , generated say by x_0, \dots, x_n . Then $R = k[tx_0, \dots, tx_n]$ where t is a transcendental, V is a projective k -variety, and since the x_i are functions on X , we get a dominating rational map $\phi: X \rightarrow V$.

Theorem 1 then says that V is normal if I^n is complete for large n , with the converse holding if for instance ϕ is birational. As a special case, we could take ϕ to be the identity map; this will be so when $X = \text{Proj } k[t_0, \dots, t_n]$ and we take I to be the subspace generated by the functions t_i/t_0 (assuming t_0 transcendental over the t_i). In view of the usual definitions, we conclude that X is normal if and only if the linear system of hypersurface sections of degree n is complete for large n .

As another example, assume X is normal, D a divisor on it, and take $I = \{f \in K \mid f \geq -D\}$. Then I is complete [2, p. 358], one calls ϕ customarily the "rational map defined by $|D|$," and we conclude

that the image of a normal variety under the rational map determined by a complete linear system is normal.

3. Application to rational singularities. We take a two-dimensional normal local domain A with maximal ideal \mathfrak{m} and algebraically closed residue field. Put $Y = \text{Spec } A$ and let p be the closed point; it is an isolated singularity of Y since A is normal. Such a singularity can be resolved by a sequence of monoidal transformations, each followed by a normalization, if needed. This process leads ultimately to a non-singular scheme \overline{W} with a proper birational morphism

$$h: \overline{W} \rightarrow W \xrightarrow{\nu} Y$$

μ ν

which factors through W , the normalized monoidal transform of Y with respect to \mathfrak{m} . Theorem 2 may now be extended to show that

$$(11) \quad h_* h^* \mathfrak{m}^n = (\mathfrak{m}^n)' \quad \text{for all } n \geq 0.$$

To see this, remark first that since W is the normalized monoidal transform, it follows from (6) that $\mathcal{G}_n = \nu^* \mathfrak{m}^n$ is an invertible sheaf of ideals. Now we claim that $\mu_* \mu^* \mathcal{G}_n = \mathcal{G}_n$ for all $n \geq 0$. For since this assertion is local on W , we may assume $\mathcal{G}_n = \mathcal{O}_W$, in which case it says, if f is in the quotient field of A , then

$$\mu^* f \in \Gamma(\overline{W}, \mathcal{O}_{\overline{W}}) \Rightarrow f \in \Gamma(W, \mathcal{O}_W).$$

This is true, for if f were not holomorphic on W , it would have a polar divisor (since W is normal), hence so would $\mu^* f$.

Putting this together with Theorem 2 proves (11), since

$$h_* h^* \mathfrak{m}^n = \nu_* \nu^* \mathfrak{m}^n = (\mathfrak{m}^n)'.$$

Suppose now that p is a rational singularity. By definition, this means that $R^1 h_* \mathcal{O}_{\overline{W}} = 0$, i.e., that the singularity contributes nothing to the arithmetic genus of the surface. In [1], Artin studies the divisor $Z = h^{-1}(p)$, and one of his key results is

$$(12) \quad \text{For all } n > 0, \text{ if } f \in A \text{ and } h^* f \geq nZ, \text{ then } f \in \mathfrak{m}^n.$$

In other words: for all $n > 0$, $h_* h^* \mathfrak{m}^n = \mathfrak{m}^n$. Comparing with (11) shows that for a rational singularity, \mathfrak{m}^n is complete for $n > 0$. Therefore by Theorem 2, its monoidal transform is normal and has only isolated singularities. Since the Leray sequence for $\overline{W} \rightarrow W \rightarrow Y$ shows that these too are also rational, it follows by induction that the rational singularity is absolutely isolated, i.e., the successive steps in the resolution process produce only isolated singularities, never multiple curves.

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