PSEUDO-UNIFORM CONVEXITY OF H^1 IN SEVERAL VARIABLES

LAURENCE D. HOFFMANN

ABSTRACT. A convergence theorem of D. J. Newman for the Hardy space H^1 is generalized to several complex variables. Specifically, in both H^1 of the polydisc and H^1 of the ball, weak convergence, together with convergence of norms, is shown to imply norm convergence. As in Newman's work, approximation of L^1 by H^1 is also considered. It is shown that every function in L^1 of the torus, (or in L^1 of the boundary of the ball), has a best H^1-approximation which, in several variables, need not be unique.

D. J. Newman [4] has shown that H^1 of the unit disc, while not uniformly convex, does have the following properties:

(i) Weak convergence, together with convergence of norms, implies norm convergence in H^1.[1]

(ii) If the distance between $k \in L^1$ and a sequence of H^1-functions tends to d, (d = distance between k and H^1), then the sequence converges in norm to the unique best H^1-approximation of k.

In this paper, (i) will be generalized to both the unit polydisc and unit ball in several variables. In fact, as in one variable, a somewhat stronger result will be obtained. On the other hand, examples will be given to show that (ii) does not generalize to these settings.

The results presented here are contained in my doctoral dissertation, written under the supervision of Professor Walter Rudin. I am most grateful for Professor Rudin's many valuable suggestions during the preparation of this paper.

Convergence theorem. Let U^N denote the unit polydisc in the space of N complex variables. The distinguished boundary of U^N is the torus, T^N. The H^1-norm in U^N is defined by

$$\|f\|_{1,N} = \sup_{0<r<1} \int_{T^N} |f(rw)| \, dm_N(w),$$

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[1] I am informed by the referee that actually this result predates Newman's paper. According to a paper by V. P. Havin [3], the theorem was first proved for the unit disc by S. Warschawski in 1930 and subsequently generalized to certain multiply connected regions by G. Ts. Tumarkin.

where dm_N denotes the Haar measure of T^N. $H^1(U^N)$ consists of those holomorphic functions on U^N whose H^1-norm is finite.

If f is any function on U^N, and $w \in T^N$, the "slice function" f_w is defined on U^1 by $f_w(\lambda) = f(\lambda w)$ $(\lambda \in U^1)$. If $h \in H^1(U^N)$, then $h_w \in H^1(U^1)$ for almost all $w \in T^N$, and, as in [6, Lemma 3.3.2], the invariance of the measure dm_N implies

$$\|h\|_{1,N} = \int_{T^N} \|h_w\|_{1,1} dm_N(w).$$

THEOREM 1. *Suppose* $f, f_n \in H^1(U^N)$ *with*

(i) $f_n \to f$ *uniformly on compact subsets of* U^N, *and*

(ii) $\|f_n\|_{1,N} \to \|f\|_{1,N}$.

Then $\|f_n - f\|_{1,N} \to 0$.

In proving this theorem for $N=1$, Newman used a factorization of H^1-functions involving Blaschke products. This technique is not applicable when $N>1$; nor is the more recent proof of C. N. Kellogg [2] in which functions in H^1 are expressed as products of H^2-functions. The proof of the theorem for $N>1$ given here applies the one-variable result to the slice functions $f_{n,w}$.

LEMMA. *Suppose* $\phi_n \geqq 0$, $\phi = \liminf \phi_n$, ϕ *and* $\phi_n \in L^1$, $\psi \leqq \phi$, *and* $\limsup \int \phi_n \leqq \int \psi$. *Then,* $\phi = \psi$ *a.e., and there exists* $n_j \to \infty$ *such that* $\phi_{n_j} \to \psi$ *a.e.*

PROOF OF LEMMA. Fatou's lemma gives the first inequality in

$$\int \phi \leqq \liminf \int \phi_n \leqq \limsup \int \phi_n \leqq \int \psi \leqq \int \phi.$$

It follows that $\phi = \psi$ a.e. and that

$$\lim \int \phi_n = \int \phi. \tag{1}$$

If $g_n = \inf\{\phi_n, \phi_{n+1}, \cdots\}$, the monotone convergence theorem gives

$$\lim \int g_n = \int \phi. \tag{2}$$

Since $g_n \leqq \phi_n$, (1) and (2) imply $\int |g_n - \phi_n| \to 0$; hence, $(g_{n_j} - \phi_{n_j}) \to 0$ a.e. for some sequence $n_j \to \infty$. But $g_n \to \phi$ a.e., so $\phi_{n_j} \to \phi$ a.e.

PROOF OF THEOREM 1. For $w \in T^N$, define

$$\phi_n(w) = \|f_{n,w}\|_{1,1}, \quad \text{and} \quad \psi(w) = \|f_w\|_{1,1}.$$

For $0<r<1$, hypothesis (i) gives

$$\int_{T^1} |f_w(r\lambda)|\, dm_1(\lambda) = \lim_{n\to\infty} \int_{T^1} |f_{n,w}(r\lambda)|\, dm_1(\lambda) \leqq \liminf_{n\to\infty} \phi_n(w),$$

so that

$$\psi(w) \leqq \liminf_{n\to\infty} \phi_n(w). \tag{3}$$

Hypothesis (ii) says that

$$\lim_{n\to\infty} \int_{T^N} \phi_n = \int_{T^N} \psi. \tag{4}$$

By the lemma, (3) and (4) imply that every sequence S_1 of positive integers contains a subsequence S_2 such that for almost all w, $\phi_n(w) \to \psi(w)$ as $n\to\infty$ in S_2. For such w, Newman's theorem asserts that

$$\|f_{n,w} - f_w\|_{1,1} \to 0 \quad \text{as } n \to \infty \quad \text{in } S_2. \tag{5}$$

We must show that

$$\int_{T^N} \|f_{n,w} - f_w\|_{1,1} dm_N(w) \to 0 \quad \text{as } n \to \infty \quad \text{in } S_2. \tag{6}$$

If $T^N = A \cup B$, this integral is majorized by

$$\int_B \|f_{n,w} - f_w\|_{1,1} dm_N(w) + \int_A \phi_n(w) dm_N(w) + \int_A \psi(w) dm_N(w). \tag{7}$$

By Egoroff's theorem and (5), A and B can be so chosen that $\|f_{n,w}-f_w\|_{1,1}\to 0$ uniformly on B as $n\to\infty$ in S_2, and so that $\int_A \psi<\epsilon$. It follows that $\phi_n\to\psi$ uniformly on B, and hence that $\int_A \phi_n<\epsilon$ for large $n\in S_2$. Hence (7) tends to zero as $n\to\infty$ in S_2. Thus every sequence S_1 contains a subsequence S_2 for which (6) holds, i.e., for which $\|f_n-f\|_{1,N}\to 0$. This completes the proof.

With only minor notational changes, this proof generalizes Newman's theorem to the unit ball B^N. In particular, if $d\nu_N$ denotes the normalized, orthogonally invariant measure on ∂B^N, the H^1-norm for B^N is defined by

$$\|f\|_1 = \sup_{0<r<1} \int_{\partial B^N} |f(rw)|\, d\nu_N(w).$$

If f is a function on B^N, and $w\in\partial B^N$, the slice function f_w is defined, as before, by $f_w(\lambda) = f(\lambda w)$ $(\lambda \in U^1)$. Moreover, the invariance of the measure $d\nu_N$ implies the basic equality

$$||h||_1 = \int_{\partial B^N} ||h_w||_{1,1} d\nu_N(w).$$

REMARK. As defined here, $H^1(U^2)$ can be identified with a closed subspace of $L^1(T^2)$: namely, the class of all L^1-functions whose Fourier transform vanishes outside the set of lattice points in the first quadrant of the plane. In [2], Kellogg shows that Newman's theorem does not generalize to the space of all functions in $L^1(T^2)$ whose Fourier transform vanishes on a certain half plane.

Best-approximation problem. If $h \in H^1(U^N)$, the radial limits $h^*(w) = \lim_{r\to 1} h(rw)$ exist for almost all $w \in T^N$. Moreover, $h^* \in L^1(T^N)$ and

$$h(z) = \int_{T^N} P(z, w) h^*(w) dm_N(w) \qquad (z \in U^N),$$

where $P(z, w)$ is the Poisson kernel in U^N. A discussion of these matters appears in [6].

Let $||f||_{1,N}$ denote the L^1-norm of $f \in L^1(T^N)$. Then,

$$||f||_{1,N} = \int_{T^N} ||f_w||_{1,1} dm_N(w),$$

where now, $f_w(\lambda) = f(\lambda w)$ for $\lambda \in T^1$.

THEOREM 2. *If $k \in L^1(T^N)$, there exists a function $h_0 \in H^1(U^N)$ for which*

$$||k - h_0^*||_{1,N} = \inf\{||k - h^*||_{1,N} : h \in H^1(U^N)\}.$$

For $N=1$, Theorem 2 is derived easily from a result of Rogosinski and Shapiro [5, Theorem 8, p. 303]. A similar argument could be given when $N>1$. However, the form of the proof given here seems somewhat more conducive to further generalization.

PROOF OF THEOREM 2. Let $d = \inf\{||k - h^*||_{1,N} : h \in H^1(U^N)\}$. There exist functions $h_n \in H^1(U^N)$, and a complex measure $d\mu$ on T^N such that $||k - h_n^*||_{1,N} \to d$, and

$$d\mu = \text{weak-star limit of } h_n^* dm_N. \tag{8}$$

For $z \in U^N$, define $h_0(z) = \int_{T^N} P(z, w) d\mu(w)$. It follows from (8) that the Fourier coefficients of h_n^* converge to those of $d\mu$. Hence h_0 is holomorphic. And since $||h_0||_{1,N} \leqq ||\mu||$, h_0 is in $H^1(U^N)$. Hence $d\mu = h_0^* dm_N$, and (8) implies $||k - h_0^*||_{1,N} \leqq \lim ||k - h_n^*||_{1,N} = d$. This completes the proof.

Theorem 2 holds with U^N and T^N replaced by B^N and ∂B^N respectively. The proof is identical to that just given, with one exception; we can no longer use Fourier coefficients to show that h_0 is holomorphic. Instead, we note that the Poisson integral representation of h_n, together with the boundedness of $\{\|h_n^*\|_{1,N}\}$, implies uniform boundedness of $\{h_n\}$ on compact sets. In addition, weak-star convergence of $h_n^* d\nu_N$ to $d\mu$ gives pointwise convergence of h_n to h_0. It follows that $h_n \to h_0$ uniformly on compact subsets of B^N. (Lars Gårding and Lars Hörmander develop the relevant properties of $H^1(B^N)$ in [1].)

THEOREM 3. *If $N>1$, there exists a function $k\in L^1(T^N)$ having infinitely many best H^1-approximations, (each of them constant).*

PROOF OF THEOREM 3. Suppose $k\in L^1(T^N)$ and

$$k(w) = k(\lambda w), \quad \text{for all } \lambda \in T^1. \tag{9}$$

Let $h\in H^1(U^N)$. For almost all w, the slice function h_w is in $H^1(U^1)$, while by (9), each k_w is constant. It follows that

$$\|k_w - h(0)\|_{1,1} = |k_w(\lambda) - h(0)| = \left|\int_{T^1} (k_w(\lambda) - h_w^*(\lambda))dm_1(\lambda)\right|,$$

and consequently,

$$\|k - h(0)\|_{1,N} \leqq \int_{T^N}\int_{T^1} |k_w(\lambda) - h_w^*(\lambda)| = \|k - h^*\|_{1,N}. \tag{10}$$

Since equality holds in (10) only when h is constant, we conclude that every best H^1-approximation of k is constant.

To construct k with infinitely many best approximations, choose a subset E of T^1 with $m_1(E)=1/2$, and let A be the set of all points $(w_1, \cdots, w_N)$ in T^N, for which $w_1w_2 \cdots w_{N-1}(\bar{w}_N)^{N-1}$ is in E. Define $k(w)$ to be 1 if $w\in A$ and -1 if $w\in T^N-A$. Notice that k satisfies (9), and that $m_N(A)=1/2$. It follows that the best H^1-approximations of k are precisely the real constants, c, for which $-1\leqq c\leqq 1$.

Except for the definition of k, this proof applies without change to the analogous theorem for the ball. To define an appropriate function $k\in L^1(\partial B^N)$, let E be the set of all $(a_1, \cdots, a_N)$ in R^N for which $\sum a_i^2=1$, $a_i\geqq 0$, and $a_1\geqq a_2$. Let A consist of those points $(a_1\lambda_1, \cdots, a_N\lambda_N)$ in ∂B^N with $(a_1, \cdots, a_N)$ in E, and $(\lambda_1, \cdots, \lambda_N)$ in T^N. Since the measure of A is 1/2, the function k which is 1 on A and -1 on ∂B^N-A has the required property.

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UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706