REGULAR-CLOSED, URYSOHN-CLOSED AND COMPLETELY HAUSDORFF-CLOSED SPACES

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ABSTRACT. Recently M. P. Berri, J. R. Porter, and R. M. Stephenson, Jr. have given a survey on P-closed and P-minimal spaces. In the present paper the first two problems of this survey will be solved: (1) The product of two Urysohn-closed spaces need not be Urysohn-closed. (2) A completely Hausdorff-closed regular space need not be regular-closed.

Given a topological property P a P-space X is called P-closed provided X is a closed set in every P-space in which it can be embedded. For instance compact Hausdorff spaces are Hausdorff-closed but not vice versa. P-closed spaces (and the closely related P-minimal spaces) have been investigated extensively especially for separation properties $P=$Hausdorff, Urysohn, regular (includes T_1). One of the major results states that P-closure (as well as P-minimality) is productive for $P=$Hausdorff. The corresponding problem for $P=$Urysohn and $P=$regular has been attacked by several authors. Scarborough and Stephenson proved independently that the product of an Urysohn-closed space and a Hausdorff-closed Urysohn space is Urysohn-closed. In §1 of the present paper we shall show that the product of two Urysohn-closed spaces need not be Urysohn-closed (Theorem 1). This settles Problem 2 of the survey of P-closed and P-minimal spaces presented recently by M. P. Berri, J. R. Porter and R. M. Stephenson, Jr. [2]. In addition in §2 we shall give a negative answer to Problem 1 of the above mentioned survey: Is a completely Hausdorff-closed regular space necessarily regular-closed? (Theorem 2.)

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1. **Products of Urysohn-closed spaces.** A topological space X is called a Urysohn-space provided that any two points of X have disjoint closed neighborhoods. It is easy to show that a Urysohn-space X is Urysohn-closed iff for any open filter F in X there exists a point

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$x \in X$ such that every closed neighborhood of x meets the closure of each member of F.

THEOREM 1. *There exist Urysohn-closed spaces Y and Z whose product $X = Z \times Y$ is not Urysohn-closed.*

PROOF. (a) *Construction of Y.* Let W be the set of all countable ordinals, and let $I = [0, 1)$ be the half-open unit interval of the reals, both sets supplied with the usual order. The product $P = W \times I$ can be ordered lexicographically and be supplied with the corresponding order topology. Let A denote the resulting space (Alexandroff's long line) and let $A' = A \cup \{a\}$ be the one-point-compactification of A. If (D_1, D_2, D_3) is a fixed partition of I into three pairwise disjoint dense subsets (with $0 \notin D_2$) then $(E_1 = W \times D_1,\ E_2 = W \times D_2,\ E_3 = (W \times D_3) \cup \{a\})$ is a partition of A' into three pairwise disjoint dense subsets. If B is the topology (= family of open sets) of A' then $B \cup \{E_2, E_3\}$ is a subbase for a new topology of A'. The resulting space Y is Urysohn-closed.

(b) *Construction of Z.* For $i = 0, 1$ let w_i denote the least ordinal with cardinality $\aleph_i$ and let W_i denote the set of all ordinals α with $\alpha \leqq w_i$ supplied with the usual order and the corresponding order topology. If we remove the point (w_1, w_0) from the product space $W_1 \times W_0$ we obtain the well-known Tychonoff plank T. Let T_1 and T_2 be two copies of T whose elements will be denoted by $(\alpha, n, 1)$ and $(\alpha, n, 2)$ respectively. For fixed $\alpha < w_1$, $n < w_0$, $i \in \{0, 1\}$, define $U_i(\alpha, n) = \{(\beta, m, i) \mid \alpha < \beta < w_1, n < m \leqq w_0\}$ and $V_i(\alpha, n) = \{(\beta, m, i) \mid \beta < \alpha, m < n\}$. In the topological union of T_1 and T_2 we identify for any $n < w_0$ the two points $(w_1, n, 1)$ and $(w_1, n, 2)$. To the resulting space Q we add a point t and define the set $\{U_1(\alpha, n) \cup \{t\} \mid \alpha < w_1, n < w_0\}$ to be a base for the neighborhoods of t. The resulting space Z is Urysohn-closed.

(c) *$X = Z \times Y$ is not Urysohn-closed.* $F_0 = \{((\alpha, n, 2), (\gamma, r)) \mid (\alpha, n, 2) \in T_2, \alpha < \gamma < w_1, r \in D_2\}$ is an open subset of X, since for any point $p = ((\alpha, n, 2), (\gamma, r))$ of F_0 we have $p \in V_2(\alpha+1, n+1) \times \{y \mid y \in E_2, (\gamma, r/2) < y\} \subset F_0$. Consequently for any $\alpha < w_1$, $n < w_0$, $F(\alpha, n) = F_0 \cap (U_2(\alpha, n) \times Y)$ is open in X and $\{F(\alpha, n) \mid \alpha < w_1, n < w_0\}$ is a base for an open filter $\mathfrak{F}$ on X. It remains to show that for each point x of X there exists a member F of $\mathfrak{F}$ and an open set U containing a closed neighborhood of x with $F \cap U = \emptyset$. This is easy to see for $x \neq (t, a)$. In case $x = (t, a)$, the set $V = (T_1 \cup \{t\}) \times (E_1 \cup E_3)$ is a closed neighborhood of x. It is sufficient to show that any point (z, y) of V has a neighborhood which does not meet F_0. This is obvious for $z \notin \mathrm{Cl}_Z T_2$ or

$y \notin \mathrm{Cl}_Y E_2$. Consequently we can assume $z=(w_1, n, 1)=(w_1, n, 2)$ and $y=(\alpha, r)$ with $r \in D_1$. But then $W=\{(\beta, n, i) \mid \alpha+1<\beta \leqq w_1, i \in \{1, 2\}\} \times \{y' \mid y' \in Y, y'<(\alpha+1,0)\}$ is a neighborhood of (z, y) which does not meet F_0 since $((\lambda, m, i), (\zeta, s)) \in W \cap F_0$ would imply $\lambda<\zeta<\alpha+1<\lambda$, which is impossible.

2. **Regular-closed and completely Hausdorff-closed spaces.** A topological space X is called completely Hausdorff provided that for any two distinct points x and y of X there exists a continuous, real-valued function $f: X \rightarrow R$ with $f(x) \neq f(y)$ (equivalently: provided there exists a continuous one-to-one map of X into a compact Hausdorff space).

THEOREM 2. *There exists a completely Hausdorff-closed regular space X which is not regular-closed.*

PROOF. Let T be the Tychonoff plank described in §1, let Z be the set of integers,and let R be the topological union of countably many copies T_i of T $(i \in Z)$ whose elements will be denoted respectively by (α, n, i). Identify in R for any $i \in Z$ and any $\alpha<w_1$ the points $(\alpha, w_0, 2i)$ and $(\alpha, w_0, 2i+1)$, and for any $i \in Z$ and any $n<w_0$ the points $(w_1, n, 2i-1)$ and $(w_1, n, 2i)$. Let Q be the corresponding quotient space of R and let βQ be its Čech-Stone-compactification. Then there exists exactly one point q in βQ such that each neighborhood of q meets each T_i. Form a new space P by replacing the point q in βQ by two different points q^+ and q^- and calling a subset U of P a neighborhood of q^+ (resp. q^-) in P iff U contains q^+ (resp. q^-) and there exists an $i \in Z$ such that $(U \backslash \{q^+\}) \cup \mathrm{Cl}_{\beta Q}(\bigcup_{j \leqq i} T_j)$ (resp. $(U \backslash \{q^-\}) \cup \mathrm{Cl}_{\beta Q}(\bigcup_{j \geqq i} T_j))$ is a neighborhood of q in βQ. It is easy to verify that P is regular using the fact that for $n \in Z$, $\mathrm{Cl}_{\beta Q}(\bigcup_{j>n} T_j) \cap \mathrm{Cl}_{\beta Q}(\bigcup_{j<n} T_j) = \{q\}$; the proof of this fact is straightforward. Consequently, the subspace X obtained from P by removing q^- is not regular-closed. But X is completely Hausdorff-closed. For let Y be a Hausdorff space which contains X as a nonclosed subspace. Then there exists a point $y \in \mathrm{Cl}_Y X - X$. Each Y-neighborhood of y meets each βQ-neighborhood of q in Q. Consequently there exists no continuous function $F: Y \rightarrow R$ with $F(y) \neq F(q^+)$. Hence Y is not completely Hausdorff.

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