ON THE CONVEXITY OF LEMNISCATES

DOROTHY BROWNE SHAFFER[1]

ABSTRACT. Let L_1 denote the lemniscate $|\prod_{\nu=1}^{n}(z-\zeta_\nu)|=1$. Assume the poles ζ_ν are inscribed in the disc $|z|\leqq a$. Let $z_0=n^{-1}\sum_{\nu=1}^{n}\zeta_\nu$. Conditions for the convexity of L_1 are established in terms of a and z_0. Sharp bounds are derived for real ζ_ν.

Let L_1 be the lemniscate $L_1: |p(z)| = \prod_{i=1}^{n}|z-\zeta_i|=1$. It was proved by Erdös, Herzog and Piranian [1] that L_1 is convex if all the ζ_i are inscribed in a disc of radius $a \leqq \sin \pi/8/(1+\sin \pi/8)$. This estimate was improved by the author [3] to $a \leqq 2^{1/2}-1=.414$. It is the object of this note to improve these bounds; a sharp result is obtained for the case of a real polynomial.

THEOREM 1. *L_1 is convex if $2^{1/2}-1\leqq a\leqq 1/3^{1/2}$ and*

$$|z_0| \leqq (1-3a^2)/(2^{3/2}a) \quad \textit{where } z_0 = n^{-1}\sum_{i=1}^{n}\zeta_i. \tag{1}$$

PROOF. The author proved [3] that any lemniscate, with its zeros inscribed in a disc of radius a is convex if it lies outside of a concentric circle of radius $2^{1/2}a$. By a lemma due to Pommerenke [2], L_1 contains the disc $|z-z_0|\leqq(1-a^2+|z_0|^2)^{1/2}$, if $a^2-|z_0|^2\leqq 1$.

It follows that L_1 lies outside the disc with center at the origin, radius

$$(1-a^2+|z_0|)^{1/2}-|z_0| \tag{2}$$

and L_1 is convex if

$$(1-a^2+|z_0|^2)^{1/2}-|z_0| \geqq 2^{1/2}a. \tag{3}$$

Inequality (3) solved for $|z_0|$ gives condition (1).

If $|z_0|=0$, i.e. the center of gravity of the zeros is assumed to be the center of the disc containing the zeros we obtain that L_1 is convex if $a\leqq 1/3^{1/2}$. If z_0 is allowed to approach the boundary, $|z_0|=a$, the previous condition $a\leqq 2^{1/2}-1$ follows.

Presented to the Society, August 28, 1969 under the title *Geometric properties of equipotential surfaces and curves*; received by the editors November 10, 1970.

AMS 1969 *subject classifications*. Primary 3010.

Key words and phrases. Lemniscate, level line, convexity.

[1] During the course of this work the author was a National Science Foundation Faculty Fellow. Publication supported by NSF grant GP-23504, Fairfield University.

THEOREM 2. *Assume in addition all the ζ_i real, then L_1 is convex if either $a \leqq 1/2$ or $1/2 \leqq a \leqq 1/2^{1/2}$ with $|z_0| \leqq (1-2a^2)/2a$.*

The proof follows from the fact that the author's proof of the convexity condition implies that a lemniscate is convex at z if the angle subtended at z by any pair of zeros is acute. It follows that for L_1 with all the zeros on a diameter of the disc, L_1 is convex if it lies outside the same disc. Applying condition (2) this will be true if $(1-a^2+|z_0|^2)^{1/2}-|z_0| \geqq a$, and the statement of the theorem follows by solution of the inequality.

These bounds are sharp, they are approached for large m by the lemniscate $|p(z)| = |(z-\frac{1}{2})^m(z+\frac{1}{2})| = 1$ and for $z_0 = 0$ by

$$|(z-2^{-1/2})(z+2^{-1/2})| = 1.$$

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK, NEW YORK 10012