SHORTER NOTES

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THE KÜNNETH FORMULA AND ABELIAN MONOIDS[1]

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ABSTRACT. A proof is given, of the Künneth formula for ordinary cohomology, using the theory of abelian monoids.

In this note, we present a proof of the Künneth formula for (ordinary) cohomology, using instead of the Eilenberg-Zilber theorem or spectral sequences, the theory of abelian monoids, (Dold-Thom, [2]), and the representation of cohomology via mappings into Eilenberg-MacLane spaces. This should be regarded as an amusing remark, rather than as a practical proof of the theorem; from this point of view, however, it becomes clear why, when one considers generalized cohomology theories, the Künneth formula must be replaced by a spectral sequence (Adams [1], and Mislin [3]). One sees again in this context, as noted many times before this, that classical cohomology, far from being ordinary, is the most singular of cohomology theories!

We deal with the category of pointed spaces, where the natural product of X and Y is the "smash product" $X\wedge Y=X\times Y/(X\times y_0)\cup(x_0\times Y)$. We prove the Künneth formula in the following form:

THEOREM. *Let X and Y be pointed* CW*-complexes, Y locally-compact. Then, for any abelian group G,*

$$\tilde{H}^n(X\wedge Y;G)\approx\bigoplus_q \tilde{H}^q(X;\tilde{H}^{n-q}(Y;G)).$$

PROOF. $\tilde{H}^n(X\wedge Y;G)\approx[X\wedge Y;K(G,n)]=[X;K(G,n)^Y]$.

By [2], $K(G,n)$ may be taken as an abelian monoid, and so clearly $K(G,n)^Y$ is also an abelian monoid. Hence, by Moore's theorem, [2, Satz 7.1], $K(G,n)^Y$ has the weak homotopy type of a product of Eilenberg-MacLane spaces. More precisely, since

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$$\pi_q(K(G,n)^Y) = [S^q; K(G,n)^Y] \approx [S^q \wedge Y; K(G,n)] = [S^q Y; K(G,n)]$$
$$\approx \tilde{H}^n(S^q Y; G) \approx \tilde{H}^{n-q}(Y; G),$$

we see that $(K(G,n)^Y$ has the weak homotopy type of

$$\times_q K(\tilde{H}^{n-q}(Y;G), q).$$

Hence

$$\tilde{H}^n(X \wedge Y; G) \approx [X; \times_q K(\tilde{H}^{n-q}(Y;G), q)]$$
$$\approx \bigoplus_q \tilde{H}^q(X; \tilde{H}^{n-q}(Y;G)).$$

REMARK. If one attempts to generalize the above proof to the case of a cohomology theory represented by an arbitrary spectrum $\{K_n\}$, one observes that it breaks down because in general K_n^Y does not decompose as a product; what presumably occurs in general is that K_n^Y can be replaced by a generalized Postnikov tower, where the fibers are products of K_n's, and this accounts for the spectral sequences, described in [1] and [3].

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