H-SPACES AND THE SUSPENSION HOMOMORPHISM

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ABSTRACT. If a space X is an H-space, then the homotopy suspension homomorphism is a monomorphism onto a direct factor in all dimensions. We present an example to show that the converse is false.

1. Introduction. Let X be an H-space and a connected CW-complex. Then the inclusion map $\iota: X \to \Omega\Sigma X$, defined by $\iota(x)(t) = (x, t) \in \Sigma X$, has a left homotopy inverse. (This was first proved for countable CW-complexes by I. M. James in [6]. For a semisimplicial argument which removes the hypothesis of countability see [4, p. 208] or, for a geometric argument, see [3].) It follows that the suspension homomorphism $\iota_*: \pi_m(X) \to \pi_m(\Omega\Sigma X)$ is a monomorphism onto a direct factor for all m. We call a space Y which satisfies this algebraic criterion (ι_* is a monomorphism onto a direct factor in all dimensions) a Σ-space and we call Y a σ-space if ι_* is a monomorphism in all dimensions. The purpose of this note is to exhibit a Σ-space which is not an H-space.

As one would expect, the example we have in mind is an infinite CW-complex and we ask whether or not there are finite CW-complexes which are Σ-spaces and not H-spaces. This question is similar to one raised by G. T. Porter: A Σ-space (in fact a σ-space) has the property that its higher order spherical Whitehead products (HOWP) vanish [8, Corollary 4]. In [9], Porter asked whether there are finite CW-complexes having trivial HOWP which are not H-spaces; to our knowledge, Porter's question is still unresolved.

In what follows we will denote inclusion maps by ι, ι' etc. and Hurewicz homomorphisms by h, h' etc.

2. The example. Consider the two-stage Postnikov system

$$K(Z_2, 2n-1) \to E_n \to K(Z, 2) \qquad (n \geqq 2)$$

with k-invariant α^n, where α is the generator of $H^2(Z, 2; Z_2)$.

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THEOREM 1. (1) *E_3 is a Σ-space.*

(2) *E_n is a σ-space for all n.*

(3) *E_n is an H-space if and only if $n=2^k$.*

Part (3) of Theorem 1 follows from the fact that the k-invariant α^n is primitive if and only if $n=2^k$ [7]. As a first step to proving parts (1) and (2) of Theorem 1, we replace E_n by an equivalent CW-complex.

Let CP^{n-1} denote the $(n-1)$-dimensional complex projective space and $p_{n-1}:S^{2n-1}\to CP^{n-1}$ the usual fibration with fiber S^1. Then $CP^n=CP^{n-1}\cup_{p_{n-1}}e^{2n}$. Starting with $CP^1=S^2$ and the Hopf map $p_1:S^3\to S^2$ we see that CP^n has a CW-structure with exactly one cell in each even dimension $\leqq 2n$. Let $d_{n-1}:S^{2n-1}\to S^{2n-1}$ be a map of degree 2 and set $X_n=CP^{n-1}\cup_{p_{n-1}d_{n-1}}e^{2n}$. Let Y_n be the space obtained from X_n by attaching m-cells, $m\geqq 2n+1$, so as to "kill" its homotopy groups in dimensions $\geqq 2n$.

LEMMA 1. *The spaces Y_n and E_n have the same homotopy type.*

PROOF. Since $p_{n-1}d_{n-1}$ represents $\pm 2\in Z=\pi_{2n-1}(CP^{n-1})$, it is clear that Y_n and E_n have the same homotopy groups and so Y_n also has a Postnikov system of the form

$$K(Z_2, 2n-1)\to E_n'\to K(Z, 2).$$

Furthermore, there is a map $Y_n\to E_n'$ which is a homotopy equivalence. Since $H^{2n}(Z, 2; Z_2)=Z_2$, the k-invariant of this Postnikov system is either 0 or α^n. If the k-invariant were 0, we would have $E_n'=K(Z, 2)\times K(Z_2, 2n-1)$, but then $H^{2n-1}(E_n'; Z_2)=Z_2$ whereas $H^{2n-1}(Y_n; Z_2)=0$, since Y_n has no $(2n-1)$-cells. We conclude that the k-invariant is α^n and that $E_n'=E_n$ which establishes the lemma.

LEMMA 2. *$\iota_*:\pi_2(Y_n)\to\pi_2(\Omega\Sigma Y_n)$ is an isomorphism.*

PROOF. This follows from the homotopy suspension theorem.

It remains to consider $\iota_*:\pi_{2n-1}(Y_n)\to\pi_{2n-1}(\Omega\Sigma Y_n)$ or equivalently, by a cellular approximation argument, $\iota_*:\pi_{2n-1}(X_n)\to\pi_{2n-1}(\Omega\Sigma X_n)$.

From the definition of X_n and CP^n we see that there is a map $f_n:X_n\to CP^n$ which maps the subspace CP^{n-1} of X_n and CP^n identically. In the following we will consider f_n to be an inclusion and, by abuse of notation, will consider the "pair" (CP^n, X_n). Set $A_n=\Omega\Sigma X_n$, $B_n=\Omega\Sigma CP^n$ and $g_n=\Omega\Sigma f_n:A_n\to B_n$.

LEMMA 3. *In the following diagram all the homomorphisms are isomorphisms and the groups are isomorphic to Z_2.*

$$\begin{array}{ccc} \pi_{2n}(CP^n, X_n) & \longrightarrow & \pi_{2n}(B_n, A_n) \\ h_1 \downarrow & & \downarrow h_2 \\ H_{2n}(CP^n, X_n) & \xrightarrow{\iota_*} & H_{2n}(B_n, A_n) \end{array}$$

PROOF. Clearly

$$\begin{aligned} H_m(X_n) = H_m(CP^n) &= Z \qquad m = 0, 2, \cdots, 2n, \\ &= 0 \qquad \text{otherwise}. \end{aligned}$$

Moreover, we may choose generators a_{2m} and b_{2m} of $H_{2m}(X_n)$ and $H_{2m}(CP^n)$, respectively, so that

$$\begin{aligned} f_{n*}(a_{2m}) &= b_{2m} \qquad 0 \leqq m \leqq n-1, \\ &= 2 \cdot b_{2n} \qquad m = n. \end{aligned}$$

Consequently, the pair (CP^n, X_n) is $(2n-1)$-connected; by the Hurewicz theorem h_1 is an isomorphism. Plainly $H_{2n}(CP^n, X_n) = Z_2$.

The Pontrjagin ring $H_*(A_n)$ ($H_*(B_n)$) is the free associative algebra on n generators, namely $\alpha_2, \cdots, \alpha_{2n}$ (resp. $\beta_2, \cdots, \beta_{2n}$), where $\alpha_{2m} = \iota_*(a_{2m})$ and $\beta_{2m} = \iota_*(b_{2m})$, $m = 1, \cdots, n$ [1]. Since the map g_n is an H-map, it follows that the induced homomorphism $g_{n*}: H_m(A_n) \to H_m(B_n)$ is an isomorphism for $m \leqq 2n-1$ and has cokernel Z_2 in dimension $2n$. As above we conclude that h_2 is an isomorphism and that $H_{2n}(B_n, A_n) = Z_2$. Finally, the homomorphism ι_* is easily seen to be an epimorphism and therefore an isomorphism. This completes the proof of the lemma.

Consider Figure 1. That the leftmost vertical homomorphism

$$\begin{array}{ccccc} & & \pi_{2n}(A_n) & \longrightarrow & H_{2n}(A_n) \\ & & \downarrow & & \downarrow g_{n*} \\ & & \pi_{2n}(B_n) & \xrightarrow{h} & H_{2n}(B_n) \\ & & \downarrow & & \downarrow j_* \\ \pi_{2n}(CP^n, X_n) & \xrightarrow{\cong} & \pi_{2n}(B_n, A_n) & \xrightarrow{\cong} & H_{2n}(B_n, A_n) \\ \downarrow \cong & & \downarrow \partial & & \downarrow \\ \pi_{2n-1}(X_n) & \xrightarrow{\iota_*} & \pi_{2n-1}(A_n) & \longrightarrow & H_{2n-1}(A_n) \\ & & \downarrow & & \\ & & \pi_{2n-1}(B_n) & & \end{array}$$

FIGURE 1

is an isomorphism follows from the fact that $\pi_{2n}(CP^n) = \pi_{2n-1}(CP^n) = 0$ and by Lemma 3 the homomorphisms in the third row are isomorphisms. It follows that ι_* is a monomorphism if and only if $j_*h=0$ and that ι_* is an epimorphism if and only if $\pi_{2n-1}(B_n)=\pi_{2n}(\Sigma CP^n)=0$.

PROOF OF PART (2) OF THEOREM 1. By the foregoing remark it suffices to show that $j_*h=0$. Now any element of $H_{2n}(B_n)$ can be expressed as $r\cdot\beta_{2n}+g_{n*}(y)$, $r=0, 1$, $y\in H_{2n}(A_n)$ (see the proof of Lemma 3). To show $j_*h=0$, it suffices to show that no element of the form $\beta_{2n}+g_{n*}(y)$ is in the image of h. Let $x\in\pi_{2n}(B_n)$ and, in order to obtain a contradiction, assume that $h(x)=\beta_{2n}+g_{n*}(y)$. Write $y=s\cdot\alpha_{2n}+z$, where z is in the subalgebra of $H_*(A_n)$ generated by the set $\{\alpha_2, \cdots, \alpha_{2n-2}\}$ and s is some integer. Consider Figure 2,

$$\begin{tikzcd}
\pi_{2n}(B_n) \arrow[r, "\sigma'"] \arrow[d, "h"] & \pi_{2n+1}(\Sigma CP^n) \arrow[r, "\iota'_*"] \arrow[d, "h'"] & \pi_{2n+1}(SU(n+1)) \arrow[d, "h''"] \\
H_{2n}(B_n) \arrow[r, "\sigma"] & H_{2n+1}(\Sigma CP^n) \arrow[r, "\iota'_*"] & H_{2n+1}(SU(n+1)) \\
H_{2n}(CP^n) \arrow[u, "\iota_*"] \arrow[ur, "\Sigma"] & &
\end{tikzcd}$$

FIGURE 2

where σ denotes the homology suspension, σ' denotes the usual bijection and ι' denotes the inclusion map. (The special unitary group $SU(n+1)$ can be given a CW-structure with ΣCP^n as a subcomplex.) From the (signed) commutativity of Figure 2 and the relations $g_{n*}(\alpha_{2n})=2\cdot\beta_{2n}$ and $\sigma(g_{n*}(z))=0$ [11, Corollary 6.2] we see that $h''\iota'_*\sigma'(x)=(2s+1)\cdot\iota'_*\Sigma(b_{2n})$. Toda [10] has shown that there is an element $\zeta\in\pi_{2n+1}(\Sigma CP^n)$ such that $\iota'_*(\zeta)$ generates $\pi_{2n+1}(SU(n+1))=Z$ and $h'(\zeta)=n!\cdot\Sigma(b_{2n})$. Therefore, there is an integer t such that $\iota'_*\sigma'(x)=t\cdot\iota'_*(\zeta)$ and we compute $h''\iota'_*\sigma'(x)=t\cdot n!\cdot\iota'_*\Sigma(b_{2n})$. Since $\iota'_*\Sigma(b_{2n})$ is of infinite order in $H_{2n+1}(SU(n+1))$ we cannot have an even multiple of this element equal to an odd multiple. This gives the desired contradiction and completes the proof of part (2) of Theorem 1.

LEMMA 4. $\pi_{2n}(\Sigma CP^n)=0$ *if and only if* $\pi_{2n}(\Sigma CP^{n-1})=Z_{n!}$.

PROOF. Consider the following exact sequence due to J. H. C. Whitehead [5, p. 115]

$$\to \pi_{2n+1}(\Sigma CP^n) \xrightarrow{h'} H_{2n+1}(\Sigma CP^n) \to \Gamma_{2n}(\Sigma CP^n)$$
$$\to \pi_{2n}(\Sigma CP^n) \to H_{2n}(\Sigma CP^n) \to .$$

It follows from the definition of Γ_{2n} that $\Gamma_{2n}=\pi_{2n}(\Sigma CP^{n-1})$. Since $H_{2n}(\Sigma CP^n)=0$, the lemma will follow if we show that $\operatorname{Im}(h')=n!\cdot Z\subset Z=H_{2n+1}(\Sigma CP^n)$. But this follows from the result of Toda cited above and the fact that h'' (see Figure 2) is a monomorphism [2].

PROOF OF PART (1) OF THEOREM 1. As noted above, it suffices to show that $\pi_6(\Sigma CP^3)=0$ or, by Lemma 4, that $\pi_6(\Sigma CP^2)=Z_6$. One can compute $\pi_6(\Sigma CP^2)=Z_6$ by considering the homotopy sequence of the pair $(\Sigma CP^2, \Sigma S^2)$ and making use of the fact that Z_6 is a subgroup of $\pi_6(\Sigma CP^2)$ (see the proof of Lemma 4).

In conclusion we point out a contrast between the Hurewicz homomorphism and the suspension homomorphism. We make no attempt to be precise here; precise statements are given in [3].

Let X^∞ denote the infinite symmetric product space of X and let $\iota: X\to X^\infty$ be the inclusion map. As is well known, the geometric condition that X be dominated by X^∞ is equivalent to the associated algebraic condition that the induced homomorphism $\iota_*:\pi_m(X)\to\pi_m(X^\infty)\cong H_m(X)$ (i.e. the Hurewicz homomorphism) be a monomorphism onto a direct factor in all dimensions.

Now let X_∞ denote the reduced product space of X and let $\iota: X\to X_\infty$ be the inclusion map. Our example shows that the geometric condition that X be dominated by X_∞ (i.e. that X be an H-space) is not equivalent to the associated algebraic condition that the induced homomorphism $\iota_*:\pi_m(X)\to\pi_m(X_\infty)\cong\pi_{m+1}(\Sigma X)$ (i.e. the suspension homomorphism) be a monomorphism onto a direct factor in all dimensions.

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