K_1 OF PROJECTIVE r-SPACE

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ABSTRACT. Let A be a commutative ring, and let X = projective r-space over A. Then we prove that K_1 of the category of locally free sheaves of finite type on X is isomorphic to the direct sum of $r+1$ copies of $K_1(A)$.

1. **Introduction.** Let A be a commutative ring, and let $X=P^r(A) = \operatorname{Proj} A[T_0, \cdots, T_r]$, where T_i are indeterminants. Let $\mathcal{V}$ be the category of locally free sheaves of finite type on X. The quasi-coherent $\mathcal{O}_X$-modules form an abelian category and $\mathcal{V}$ is an admissible subcategory. Then the groups $K_0(\mathcal{V})$ and $K_1(\mathcal{V})$ as well as the automorphism category $\sum\mathcal{V}$ are defined as in Chapter VIII of [1].

Let $\mathcal{P}(A)$ be the category of projective A-modules of finite type. Then there are group homomorphisms $K_0(A)\xrightarrow{h_n}K_0(P^r(A))$ induced by $P\to P\otimes_A\mathcal{O}(n)$, $P\in\mathcal{P}(A)$. It is proved in SGA 6, Exposé 6, Theorem 1.1 that the h_i, $0\leqq i\leqq r$, set up an isomorphism $\bigoplus_{i=0}^r K_0(A) \cong K_0(\mathcal{V})$.

It is the aim of this paper to prove a corresponding result for $K_1(\mathcal{V})$. That is, the homomorphisms $K_1(A)\xrightarrow{h_n}K_1(\mathcal{V})$ induced by $(P, \alpha) \to(P\otimes_A\mathcal{O}(n), \alpha\otimes 1)$ set up an isomorphism $\bigoplus_{i=0}^r K_1(A)\cong K_1(\mathcal{V})$. $(P\in\mathcal{P}(A), \alpha\in\operatorname{Aut}_A(P)$.)

2. **Preliminaries.** The definitions and basic properties of Proj can be found in [2, §2]. Some properties we will use are the following: Let $f:X\to\operatorname{Spec} A$ be the structure morphism. Let $B=A[T_0, \cdots, T_r]$, and let $\mathcal{B}$ be the category of graded B-modules, the morphisms being B-linear maps of degree 0. Let $\mathcal{C}$ be the category of quasi-coherent $\mathcal{O}_X$-modules. Then $\sim$ is an exact functor from $\mathcal{B}$ to $\mathcal{C}$, and the functor $\Gamma_*:\mathcal{C}\to\mathcal{B}$ is defined by $\Gamma_*(F) = \bigoplus_{n\in Z}\Gamma(X, F(n))$.

Then we have $\sim\cdot\,\Gamma_*=1_{\mathcal{C}}$. Furthermore $(\bigoplus_{n\in Z} M_n)^\sim$ depends only on $\bigoplus_{n\geq n_0} M_n$, for any n_0. Also $\mathcal{O}(n)=B(n)^\sim$, where $B(n)_m=B_{m+n}$.

If $F\in\mathcal{C}$, $P\in\mathcal{P}(A)$, then by $P\otimes_A F$, I mean $f^*(P)\otimes_{\mathcal{O}_X} F$. Then if $M\in\mathcal{B}$, $(P\otimes_A M)^\sim=P\otimes_A M^\sim$.

We also have the following, which follow from Propositions 1.3 and 1.6 of SGA 6, Exposé 6 respectively:

THEOREM A. *If V is a locally free $\mathcal{O}_X$-module of finite type, then $\exists$ an integer n_0 such that for $n\geqq n_0$:*

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(i) $R^i f_*(V(n))=0$ *for* $i\geqq 1$,

(ii) $f_*(V(n))=\Gamma(V(n))$ *is in* $\mathcal{P}(A)$,

(iii) $\bigoplus_{n\geq n_0} f_*(V(n))= \bigoplus_{n\geq n_0}\Gamma(V(n))$ *is a graded B-module of finite presentation.*

THEOREM B. *Let M be a graded B-module of finite presentation, flat over A. Then there exists a resolution $0\to L_{r+1}\to\cdots\to L_0\to M\to 0$ where the L_i are graded projective B-modules of finite type.*

3. **Proof that the h_i generate $K_1(\mathcal{V})$.** Let $V\in\operatorname{obj}\mathcal{V}$, and let α be an automorphism of V. Then by Theorem A, $\exists\, n_0$ such that $M=\bigoplus_{n\geq n_0}\Gamma(V(n))$ is a graded B-module of finite presentation. $\Gamma_*(\alpha)$ gives an automorphism of M in the category $\mathcal{B}$. By Theorem A (ii), $\Gamma(V(n))$ is a projective A-module of finite type. Hence Theorem B applies, to give a resolution

$$0\to L'_{r+1}\to L'_r\to\cdots\to L'_0\to M\to 0$$

of M by graded projective B-modules of finite type.

Let $\mathcal{H}(B)$ be the category of (nongraded) B-modules that have a resolution of finite length by (nongraded) projective B-modules of finite type. L'_0 is projective in $\mathcal{B}$, so by Proposition 4.5(a), Chapter VIII of [1], α lifts to an automorphism α_0 (in $\mathcal{B}$) of $L_0=L'_0\oplus L'_0$. We then have an exact sequence $0\to N_0\to L_0\to M\to 0$. $N_0\in\mathcal{H}(B)$ by Proposition 6.3, Chapter III of [1]. Thus $(N_0, \alpha_0|N_0)\in\sum\mathcal{B}$. There exists a graded projective B-module of finite type mapping onto N_0, so we can repeat this process, to obtain a resolution

$$0\to(L_{r+1},\alpha_{r+1})\to(L_r,\alpha_r)\to\cdots\to(L_0,\alpha_0)\to(M,\alpha)\to 0$$

of (M,α) in $\sum\mathcal{B}$, where L_i is a graded projective B-module of finite type. We can terminate the resolution with L_{r+1} because Theorem B implies that M has homological dimension $\leqq r+1$.

Let $I=(T_0,\cdots,T_r)\subset B$. Then $L_i/IL_i=\bar{L}_i$ is a graded projective A-module of finite type, and by Proposition 3.3, Chapter XII of [1], $\bar{L}_i\otimes_A B\cong L_i$ as graded B-modules.

If we write $\bar{L}_i\cong\bigoplus_j\bar{L}_{ij}$, where $\bar{L}_{ij}$ is of degree j, then $L_i\cong\bigoplus_j\bar{L}_{ij}\otimes_A B=\bigoplus_j L_{ij}$, where $L_{ij}=\bar{L}_{ij}\otimes_A B$.

If $j_1<j_2$, there are no nonzero morphisms in $\mathcal{B}$ from L_{ij_1} to L_{ij_2}, since such a morphism is determined by the image of $\bar{L}_{ij_1}\otimes_A A$, and the degree j_1 component of L_{ij_2} is zero. Therefore the matrix representing α_i in the above direct sum decomposition of L_i is upper triangular. Let α_{ij} be the diagonal entries of this matrix, where $\alpha_{ij}: L_{ij}\to L_{ij}$.

All $\mathcal{B}$-homomorphisms $\alpha_{ij}: L_{ij}\to L_{ij}$ are of the form $\bar{\alpha}_{ij}\otimes 1$, where $\bar{\alpha}_{ij}\in\operatorname{Hom}_A(\bar{L}_{ij},\bar{L}_{ij})$.

If we apply the functor $\tilde{}$ we get a resolution

$$0 \to (\tilde{L}_{r+1}, \tilde{\alpha}_{r+1}) \to (\tilde{L}_r, \tilde{\alpha}_r) \to \cdots \to (\tilde{L}_0, \tilde{\alpha}_0) \to (V, \alpha) \to 0$$

in $\sum \mathcal{V}$.

Thus $k_1(V, \alpha) = \sum_{i=0}^{r+1} (-1)^i k_1(\tilde{L}_i, \tilde{\alpha}_i)$, where $k_1(V, \alpha)$ denotes the image in $K_1(\mathcal{V})$ of $(V, \alpha) \in \sum \mathcal{V}$.

Because the matrices for the $\tilde{\alpha}_i$ are triangular, we have

$$k_1(\tilde{L}_i, \tilde{\alpha}_i) = \sum_j k_1(\tilde{L}_{ij}, \tilde{\alpha}_{ij}).$$

But $\tilde{L}_{ij} = \overline{L}_{ij} \otimes_A (\mathcal{O}(-j))$ (where $\overline{L}_{ij}$ is now regarded as being of degree 0 as an A-module). So

$$k_1(\tilde{L}_i, \tilde{\alpha}_i) = \sum_j k_1(\overline{L}_{ij} \otimes_A \mathcal{O}(-j), \overline{\alpha}_{ij} \otimes 1).$$

Thus the images of $h_n : K_1(A) \to K_1(\mathcal{V})$, where h_n is given by $h_n(M, \alpha) = (M \otimes_A \mathcal{O}(n), \alpha \otimes 1)$ generate $K_1(\mathcal{V})$ $(M \in \mathcal{P}(A))$.

4. Proof that the h_i, $0 \leqq i \leqq r$, generate $K_1(\mathcal{V})$. Since $T_0, \cdots, T_r$ form a B-sequence, the Koszul complex gives an exact sequence

$$0 \to B^{\binom{r+1}{r+1}} \to \cdots \to B^{\binom{r+1}{i}} \to \cdots \to B^{r+1} \to B \to A \to 0.$$

The maps in the sequence (except for the map $B \to A$) are of degree 1. If we shift the gradings so as to make the maps of degree 0, and apply the functor $\tilde{}$, we get exact sequences in $\mathcal{V}$

$$0 \to \mathcal{O}(n) \to \mathcal{O}(n+1)^{r+1} \to \mathcal{O}(n+2)^{\binom{r+1}{2}} \to \cdots \to \mathcal{O}(r+n+1) \to 0$$

for all n. If $k_1(A^m, \alpha)$ is an arbitrary element of $K_1(A)$, then we get an exact sequence

$$0 \to (A^m \otimes_A \mathcal{O}(n), \alpha \otimes 1) \to (A^m \otimes_A \mathcal{O}(n+1)^{r+1}, \alpha \otimes 1)$$
$$\to \cdots \to (A^m \otimes_A \mathcal{O}(n+r+1), \alpha \otimes 1) \to 0.$$

Hence we have the relations

$$\sum_{i=0}^{r+1} (-1)^i \binom{r+1}{i} h_{n+i}, \qquad \forall n.$$

Thus the images of h_i, $0 \leqq i \leqq r$, generate $K_1(\mathcal{V})$. (The same relations hold in the K_0-case, as is shown in SGA 6, Exposé 6.)

5. Independence of the h_i, $0 \leqq i \leqq r$. Every element in $K_1(A)$ is of the form $k_1(A^r, \alpha)$, hence every element in $h_i K_1(A)$ is of the form $k_1(\mathcal{O}(i)^{r_i}, \alpha_i \otimes 1)$. Suppose we have a relation in $K_1(\mathcal{V})$ of the form

$$\sum_{i=0}^{r} h_i(x_i) = \sum_{i=0}^{r} k_1(\mathcal{O}(i)^{r_i}, \alpha_i \otimes 1) = 0$$

where $x_i = k_1(A^{r_i}, \alpha_i) \in K_1(A)$, $\alpha_i \in \mathrm{Aut}_A(A^{r_i})$.

This can be written in the form $K_1(F, \alpha) = 0$ where $F = \bigoplus_{i=0}^{r} \mathcal{O}(i)^{r_i}$, $\alpha = \bigoplus (\alpha_i \otimes 1)$.

Then we also have $\sum_{i=0}^{r} k_1(\mathcal{O}(i+n)^{r_i}, \alpha_i \otimes_A 1_{\mathcal{O}(i+n)}) = 0$ for all n.

$k_1(F, \alpha) = 0$ means that there exist a finite number of exact sequences

$$(1) \qquad \begin{array}{l} 0 \to (G_i, \gamma_i) \to (H_i, \delta_i) \to (I_i, \sigma_i) \to 0, \\ 0 \to (G_j', \gamma_j') \to (H_j', \delta_j') \to (I_j', \sigma_j') \to 0 \end{array}$$

in $\sum \mathcal{V}$, and objects M_i, N_j with automorphisms α_i, β_i; α_j', β_j' respectively, such that in the free abelian group on isomorphism classes of objects in $\sum \mathcal{V}$, we have

$$\begin{aligned} (F, \alpha) = & \sum_i [(G_i, \gamma_i) + (I_i, \sigma_i) - (H_i, \delta_i)] \\ & + \sum_j [(H_j', \delta_j') - (G_j', \gamma_j') - (I_j', \delta_j')] \\ & + \sum_i [(M_i, \alpha_i\beta_i) - (M_i, \alpha_i) - (M_i, \beta_i)] \\ & + \sum_j [(N_j, \alpha_j') + (N_j, \beta_j') - (N_j, \alpha_j'\beta_j')]. \end{aligned}$$

Corresponding relations are obtained for $(F(n), \alpha(n))$ by tensoring with $\mathcal{O}(n)$.

By Theorem A (i), there exists an n_0 such that if $n \geqq n_0$, all the exact sequences (1) (after tensoring with $\mathcal{O}(n)$) remain exact after applying the functor f_*.

Hence for $n \geqq n_0$, we have

$$k_1(f_*F(n), f_*\alpha(n)) = 0 \quad \text{in } K_1(A).$$

But

$$f_*(\mathcal{O}(i)^{r_i}, \alpha_i \otimes 1_{\mathcal{O}(i)}) = (A^{r_i}, \alpha_i) \otimes_A f_*(\mathcal{O}(i)) = (A^{r_i}, \alpha_i) \otimes_A A^{\sigma_i},$$

where σ_i = rank over A of the ith graded component of B. Thus we have relations in $K_1(A)$

$$(2) \qquad \sum_{i=0}^{r} \sigma_{n+i} x_i = 0, \qquad n \geqq n_0, \quad \text{where } x_i = k_1(A^{r_i}, \alpha_i) \in K_1(A).$$

The rest of the proof is now identical with that in SGA 6, Exposé 6, for the K_0 case. We have:

$$\sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}\sigma_{p-i}=0,\qquad p\geqq 1,\tag{3}$$

where $\sigma_k=0$ by definition, if $k<0$. This follows from the exactness of the Koszul complex. Using (3) we get

$$\sum_{j=0}^{r}\left(\sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}\sigma_{n_0+r+j-i}\right)x_j$$
$$=\sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}\left(\sum_{j=0}^{r}\sigma_{n_0+r+j-i}x_j\right)=0.$$

If $i\neq r+1$, $n_0+r-i\geqq n_0$, so by (2) we have $\sum_{j=0}^{r}\sigma_{n_0+r+j-i}x_j=0$.

We are left with, for $i=r+1$,

$$\sum_{j=0}^{r}\sigma_{n_0-1+j}x_j=0.$$

This is (2) for $n=n_0-1$. We can continue down by induction to $n=-r$, yielding $\sigma_0x_r=0$. But $\sigma_0=1$, so $x_r=0$.

For $n=-r+1$, (2) says $\sigma_0x_{r-1}+\sigma_1x_r=0$. $\therefore$ $x_{r-1}=0$.

In a similar way we get all the $x_i=0$. Thus the relation $\sum_{i=0}^{r}h_i(x_i)=0\Rightarrow x_i=0$, $0\leqq i\leqq r$. Thus we have proved:

THEOREM. *Let A be a commutative ring, and $\mathcal{V}$ the category of locally free sheaves on $P^r(A)$. Then the homomorphisms $h_i:K_1(A)\to K_1(\mathcal{V})$ induced by $(P^r,\alpha)\to(P\otimes_A\mathcal{O}(n),\alpha\otimes 1)$ satisfy the relations*

$$\sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}h_{i+n}=0$$

for all n, and $h_0,\cdots,h_r$ set up an isomorphism $\bigoplus_{i=0}^{r}K_1(A)\cong K_1(\mathcal{V})$.

6. **Further remarks.** (1) The result for K_0 is proved in SGA 6, Exposé 6, with more general schemes in place of Spec A, but I do not know how to give the proof for K_1, except with an affine base scheme.

(2) $K_1(A)$, $K_0(\mathcal{V})$ and $K_1(\mathcal{V})$ are all $K_0(A)$-modules, and the theorem can be stated in the form

$$K_0(\mathcal{V})\otimes_{K_0(A)}K_1(A)\simeq K_1(\mathcal{V}).$$

In this form it generalizes the result of [4] for the case of projective r-space over an algebraically closed field.

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