ON THE NONEXISTENCE OF SLICES FOR
p-ADIC TRANSFORMATION GROUPS

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ABSTRACT. If a compact Lie group acts effectively on a manifold, then local cross-sections to the orbit structure can be constructed. They are called *slices*. We prove that if a compact group acts effectively on a manifold, but it is not a Lie group, then a slice cannot exist at every point.

Introduction. Recent papers of R. F. Williams have focused new attention on the *Hilbert-Smith conjecture:* If a compact group G acts effectively on a manifold, then G is a Lie group. Much interest has centered around properties which arise if one assumes that the conjecture is false, and that the p-adic group A_p acts effectively on a manifold M. The purpose of this paper is to show that if such an action occurs, then for any reasonable definition of a slice, there is at least one point on the manifold at which a slice fails to exist.

DEFINITION. Let (X, G) be a compact G-space, and let $x \in X$. A *slice at x* is a subset Y of X containing x and satisfying:

(i) Whenever $g \in G$ and $y \in Y$, $g(y) \in Y \Leftrightarrow g \in G_x$.

(ii) If Q is an open neighborhood of $e \in G/G_x$ which serves as a local cross-section to the cosets of G_x at the identity of G, then $(g, y) \rightarrow g(y)$ defines a homeomorphism of $Q \times Y$ onto $Q(Y)$. Moreover, $Q(Y)$ is open in X.

REMARK. Equivalent definitions of a slice for the case where G is a compact Lie group, may be found in Montgomery and Yang [4], and in Chu [1], and Palais [6], for noncompact Lie groups.

DEFINITION. Let G_k denote a cyclic group of order p^k, and let $g_k \in G_k$ be a fixed generator. For each pair of integers $i, j \geq 0$ with $i \geq j$ we let $\phi_{ij}: G_i \rightarrow G_j$ be the homomorphism which arises by sending $g_i \rightarrow g_j$. Note that ϕ_{ii} is the identity map on G_i. With the positive integers as directed indexing set, it is easily verified that the collection of groups and homomorphisms $\{G_i, \phi_{ij}\}$ forms an inverse system. The group $A_p = \lim_{\leftarrow} \{G_i\}$ is called the *p-adic group*. Its topology and group operation are those which are induced from the product $\mathsf{X}_i G_i$, and from the standard theorems on inverse limits we find that A_p is abelian, Hausdorff, and compact.

We can obtain a large number of facts about A_p by constructing it in a different way. In particular we can examine its algebra and topology in more depth. We proceed as follows; consider A_p as the set of all sequences

$$(a_0, a_1, a_2, \cdots) \quad \text{where} \quad a_j \in \{0, 1, 2, \cdots, p-1\}.$$

The group operation is performed by adding two sequences coordinate-wise modulo p, from left to right, with the carry-over of a one to the next coordinate whenever the sum is greater than or equal to p. The identity $e \in A_p$ is the sequence $(0, 0, 0, \cdots)$, and we define a system of neighborhoods $\{U_n\}$ of the identity:

$$U_n = \{(a_j) : a_j = 0 \text{ for } j < n\}.$$

With this topology we find that A_p is totally disconnected, zero-dimensional, perfect, and homeomorphic to the Cantor set.

REMARK. The U_n are obviously closed subgroups of A_p, each isomorphic to A_p. The fact that p is prime is used in an easy proof that the U_n are the only closed, nontrivial subgroups of A_p. The demonstration of the equivalence of the two definitions is a routine exercise in inverse limits.

THEOREM. *If the p-adic group acts effectively on a manifold M, then there is at least one point $X \in M$ at which a slice fails to exist.*

We will prove this theorem by a sequence of lemmas, the first of which covers the case where the given action is free. The crucial point centers around finding a point with trivial isotropy group, all points being such if the action is free.

LEMMA 1. *Let $G = A_p$ (the p-adic group) act on the locally-connected space X. Let $x \in X$ be a point for which the isotropy group G_x is trivial. Then there does not exist a slice at x.*

PROOF. We suppose that Y is a slice at x. Since $G/G_x = G$, we are given an open neighborhood Q of the identity, for which the map $Q \times Y \to Q(Y) = U$ defined above is a homeomorphism onto an open neighborhood of x in X. Since each such open neighborhood Q is a closed subgroup of G, isomorphic to G, we may assume that $Q = G$. Thus we have a homeomorphism $f: G \times Y \to U$. We let V be the connected component of U containing x, and since X is locally connected, we have that V is open in X. The set $f^{-1}(V)$ is open in $G \times X$ and connected. But since G is perfect and totally disconnected, there are no open connected subsets of $G \times X$, which is a contradiction. □

Our purpose now is to show that for any effective action of the p-adic group on a manifold, there always exists a point which

satisfies the hypothesis of Lemma 1. An immediate application of Newman's theorem (cf. Montgomery, Samelson and Zippin [3]) gives the result that if a compact abelian Lie group acts effectively on a manifold, then the principal isotropy group is trivial. For a definition of principal isotropy group, and its associated ideas, see Montgomery, Samelson and Yang [2]. From observing that for any action of the p-adic on a space X, either $G_x \subset G_y$ or $G_y \subset G_x$ for any two points $x, y \in X$, we adopt the following:

DEFINITION. If the p-adic group G acts effectively on a space X, then by a principal point for the action we mean a point $x \in X$ such that $G_x \subset G_y$ for all $y \in X$.

LEMMA 2. *If the p-adic group acts effectively on a manifold M, then there is at least one point $x \in M$ which is a principal point for the action. Furthermore, G_x is trivial.*

PROOF. We shall prove that there is a point $x \in M$ for which G_x is trivial. We thus suppose that for each point $x \in M$, G_x is nontrivial. We let $g \in G$ be not the identity.

If $H \subset G$ is any nontrivial closed subgroup of G, then some power of g belongs to H. This is easily seen from our remark following the second definition of the p-adic group. Since if we take the closed subgroup U_k, then $(U_k)^p = U_{k+1}$. Thus, for each point $x \in M$, some power of g belongs to G_x. The map $g: M \to M$ is then a pointwise periodic homeomorphism, since some power of g belongs to every nontrivial closed subgroup of G. By a theorem of Montgomery and Zippin [5], g is periodic. But since G has no elements of finite order, this contradicts the fact that the action was assumed effective.   □

It is now easily seen that the theorem as stated is an easy consequence of Lemmas 1 and 2, in light of the fact that manifolds are locally connected.

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