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TOPOLOGICAL CHARACTERIZATION OF PSEUDO-$\aleph$-COMPACT SPACES

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ABSTRACT. Pseudo-$\aleph$-compact spaces, usually defined by cardinal bounds on uniform coverings of the fine uniformity, are characterized by cardinal bounds on locally finite open (resp. closed) coverings. This yields a new characterization of pseudocompact spaces as well as of G. Aquaro's paralindelöfian spaces.

According to [4, p. 135], a *pseudo-$\aleph$-compact space*, $\aleph$ an infinite cardinal, is a uniformizable space whose fine uniformity has covering character $\leqq\aleph$—which means that every uniform covering of the fine uniformity (=finest compatible uniformity) has a uniform refinement of cardinality $<\aleph$. For $\aleph=\aleph_0$, we have the well-known pseudocompact spaces; for $\aleph=\aleph_1$, we have the *paralindelöfian spaces* of [1]. Although many topological characterizations of pseudocompact spaces are well known, there is no topological characterization of pseudo-$\aleph$-compact spaces with $\aleph>\aleph_0$, since in [1] and [4, Chapter vii] they are investigated only from the viewpoint of uniform spaces (roughly speaking, [1, Proposition 3] characterizes paralindelöfian spaces by $\mathfrak{U}$-reducible locally finite open coverings, but this characterization—a part of the very complicated notion of "$\mathfrak{U}$-reducible"—is not purely topological since $\mathfrak{U}$-reducible locally finite open coverings are exactly the elements of a base for the fine uniformity).

In this note, pseudo-$\aleph$-compact spaces are characterized by cardinal bounds on locally finite open or closed coverings. This should be new also for pseudocompact spaces since no cardinal bounds are assumed, a priori, on coverings.

TERMINOLOGY. Our terminology is based on [4], with the sole exception that our spaces are not assumed, a priori, to be Hausdorff.

THEOREM. *For a uniformizable space X the following statements are pairwise equivalent:*

(1) *X is pseudo-$\aleph$-compact.*

(2) *Every locally finite disjoint collection of nonempty open sets of X has cardinality $<\aleph$.*

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(3) *Every locally finite closed covering of X has a subcovering of cardinality* $<\aleph$.

(4) *Every locally finite open covering* $\mathcal{U}$ *of X has a subcollection* $\mathcal{V}$ *such that* $\mathrm{Card}(\mathcal{V})<\aleph$ *and* $X=\bigcup_{V\in\mathcal{V}}\overline{V}$.

PROOF. (1)→(2). Let $\mathcal{U}$ be a locally finite disjoint collection of nonempty open sets of X. In each $U\in\mathcal{U}$ choose a point x_U. Since X is uniformizable, for every $U\in\mathcal{U}$ there is a continuous map $f_U:X\to[0,1]$ which is 1 on x_U and 0 on $X\setminus U$. Let $e_U=(y_V)_{V\in\mathcal{U}}$ be the element of the Hilbert space $l_2(\mathcal{U})$—see [3, Chapter ix, §8] for its definition—defined by $y_U=1$ and $y_V=0$ for $V\neq U$. Since $\mathcal{U}$ is locally finite,

$$f(x) = \sum_{U\in\mathcal{U}} f_U(x)e_U$$

defines a continuous map $f:X\to l_2(\mathcal{U})$. Let μ be the fine uniformity of X. By a well-known theorem, [4, i.21], $f:\mu X\to l_2(\mathcal{U})$ is uniformly continuous. Therefore there is a uniform cover $\mathcal{V}\in\mu$ such that

$$\mathrm{diam}(f(V)) < 1/2 \qquad (V\in\mathcal{V}).$$

By (1), $\mathcal{V}$ has a uniform refinement $\mathcal{W}$ of cardinality $<\aleph$. For every $U\in\mathcal{U}$ there is $W_U\in\mathcal{W}$ such that $x_U\in W_U$. Let us show that $U'\neq U''$ implies $W_{U'}\neq W_{U''}$. For, $W_{U'}=W_{U''}$ implies $x_{U'}$ and $x_{U''}\in W_{U'}$, so that

$$\|f(x_{U'}) - f(x_{U''})\| < 1/2.$$

But this is impossible since $U_0\neq U$ implies $f_{U_0}(x_U)=0$, so that $f(x_{U'})=e_{U'}$ and $f(x_{U''})=e_{U''}$, and $\|e_{U'}-e_{U''}\|=\sqrt{2}$. Hence $U\rightsquigarrow W_U$ is a $1-1$ map $\mathcal{U}\to\mathcal{W}$, so that $\mathrm{Card}(\mathcal{U})\leqq\mathrm{Card}(\mathcal{W})<\aleph$.

(2)→(3). Let $(C_\alpha)_\alpha$ be a locally finite closed covering of X. By [3, p. 177, Exercises 7 and 6], there is a disjoint family $(U_\alpha)_\alpha$ of open sets of X such that $\overline{U}_\alpha\subseteq C_\alpha$ for every α, and $X=\bigcup_\alpha\overline{U}_\alpha$. (We recall briefly the construction: Let $\leqq$ be a well-order for the indexing set. One may show inductively that

$$\{\overline{\mathrm{Int}(C_\beta)}\mid\beta\leqq\alpha\}\cup\{C_\beta\mid\beta>\alpha\}$$

is a covering of X for every α. Consequently it is easily seen that the sets

$$U_\alpha = \mathrm{Int}(C_\alpha)\setminus\bigcup_{\beta<\alpha}\overline{\mathrm{Int}(C_\beta)}$$

form the desired family.) Then obviously (3) follows from (2).

(3)→(4). Because $\{\overline{U}\mid U\in\mathcal{U}\}$ is a locally finite closed covering of X.

(4)$\rightarrow$(1). Let $\mathfrak{U}$ be a uniform cover of the fine uniformity μ of X. It follows from [4, i.14] that $\mathfrak{U}$ has a closed uniform refinement $\mathfrak{V}$. By [4, vii. 4 and i.19] or, alternatively, [2, §3], $\mathfrak{V}$ has an open locally finite uniform refinement $\mathfrak{W}$. By (4), there is $\mathfrak{W}_0 \subseteq \mathfrak{W}$ such that $X = \bigcup_{W \in \mathfrak{W}_0} \overline{W}$ and $\mathrm{Card}(\mathfrak{W}_0) < \aleph$, so that $\mathfrak{U}$ has a refinement of cardinality $<\aleph$. Hence X is pseudo $\aleph$-compact by [4, ii.33]. q.e.d.

As pointed out by [1], the one-point compactification of a discrete uncountable space cannot have Souslin property, so that local finiteness cannot be relaxed in the above theorem. Since any uniform cover of the fine uniformity is refined by a σ-uniformly discrete covering by [2, §3] or, alternatively, [4, vii.4 and i.19], we may change "locally finite" with "discrete" in statement (2) of the theorem.

The following result is well known for pseudocompact spaces.

COROLLARY. *If U is an open subset of a pseudo-$\aleph$-compact space, then $\overline{U}$ is pseudo-$\aleph$-compact.*

PROOF. Let $(C_\alpha)_{\alpha \in A}$ be a closed covering of $\overline{U}$ which is locally finite in the subspace $\overline{U}$ of X. Clearly $\{C_\alpha \mid \alpha \in A\} \cup \{X \backslash U\}$ is a locally finite closed covering of X, so that it has a subcovering of cardinality $<\aleph$ by the theorem. Therefore there is $A' \subseteq A$ such that $\mathrm{Card}(A') < \aleph$ and $U \subseteq \bigcup_{\alpha \in A'} C_\alpha$. Since $(C_\alpha)_\alpha$ is locally finite and closed in $\overline{U}$,

$$\overline{U} \subseteq \overline{\bigcup_{\alpha \in A'} C_\alpha} = \bigcup_{\alpha \in A'} C_\alpha. \qquad \text{q.e.d.}$$

Define X *locally pseudo-$\aleph$-compact* iff X is uniformizable and every point of X has a neighborhood which is a pseudo-$\aleph$-compact subspace of X. From the preceding corollary it follows that in a locally pseudo-$\aleph$-compact space every point has a neighborhood base made of pseudo-$\aleph$-compact subspaces. Let X be a locally pseudo-$\aleph$-compact space, $\mathcal{P}$ the collection of all its pseudo-$\aleph$-compact subspaces, ∞ a point not in X, and $Y = X \cup \{\infty\}$. Let τ_0 be the topology on Y having as basis

$$\{U \subseteq X \mid U \textit{ is open}\} \cup \{Y \backslash P \mid P \in \mathcal{P}\}.$$

Let τ be the weak topology induced on Y by all continuous functions $(Y, \tau_0) \rightarrow \boldsymbol{R}$. One can derive from the theorem that (Y, τ) is pseudo-$\aleph$-compact. Moreover, $\mathrm{id}: X \rightarrow (Y, \tau)$ is a homeomorphism into. This space (Y, τ) may be looked at the one-point pseudo-$\aleph$-compactification of X. The key properties of one-point pseudo-$\aleph$-compactifications are like those of one-point compactifications, with "pseudo-$\aleph$-compact" in place of "compact".

Finally, note that every open subset of an arbitrary product of uniformizable spaces, each with density character $\leqq \aleph_\alpha$, is pseudo-$\aleph_{\alpha+1}$-compact. For, every uniform covering is realized by a continuous map into a metric space, and such a map may be factored (as one may check by repeating the proof of Gleason factorization theorem [**4**, vii.23]) as $g \circ \pi_C$ with π_C a countable projection and g continuous.

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