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COMMENTS ON THE CONTINUITY OF DISTRIBUTION FUNCTIONS OBTAINED BY SUPERPOSITION[1]

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ABSTRACT. Let $\{X(t)\}$ be a differential process and Y a nonnegative random variable independent of the process. We consider whether the superposition $X(Y)$ can have a continuous probability distribution. If the process has continuous distributions, then the superposition is continuous if and only if $P[Y=0]=0$. If the process has discontinuous distributions and no trend, then no superposition can have continuous distribution. If the process has discontinuous distributions and nonzero trend, then the superposition onto a random epoch has continuous distribution if and only if Y has continuous distribution.

1. **Introduction.** Our terminology is, in general, that of [**4**]. Suppose that $Y, X_1, X_2, \cdots$ are independent random variables with $\mathfrak{L}(Y) = \mathcal{P}(\lambda)$ and that $X_1, X_2, \cdots$ are identically distributed. Clearly, the distribution function of the random sum $Z=X_1+\cdots+X_Y$ will have a jump discontinuity of size greater than or equal to $P[Y=0] = e^{-\lambda}$ at the origin. A *differential process* $\{X(t)/t\in[0, \infty)\}$ is a stochastic process with stationary, independent increments that is continuous in law and satisfies the initial condition $P[X(0)=0]=1$. Note that if Y is independent of $\{X(t)\}$, then the superposition $X(Y)$ is a random sum as described above. We shall consider such superpositions when Y is not necessarily integer-valued and concern ourself with the question of whether or not the superposition has continuous distribution.

Every differential process is an infinitely divisible process; that is, the characteristic functions of the process are of the form

$$f_{X(t)}(u) = \exp\left\{t\left(i\gamma_X u - \sigma_X^2 u^2/2 + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)dM_X(x)\right)\right\},$$

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where γ_X, σ_X^2, and M_X are the Lévy parameters uniquely associated with the given distribution. The Lévy spectral function M_X is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$, is asymptotically zero $(M_X(-\infty) = 0 = M_X(\infty))$, and satisfies the integrability condition

$$\int_{-1}^{-0} + \int_{+0}^{+1} x^2 dM_X(x) < \infty.$$

P. Hartman and A. Wintner in [2] and J. R. Blum and M. Rosenblatt in [1] have proved that a necessary and sufficient condition for an infinitely divisible random variable to have continuous distribution is that $\int_{-\infty}^{\infty} dM(x) = \infty$ or $\sigma^2 > 0$. Consequently, if the $\{X(t)\}$ process has discontinuous distributions, then its characteristic functions can be written in the form

$$f_{X(t)}(u) = \exp\left\{t\left(i\tau_X u + \int_{-\infty}^{\infty} (e^{iux} - 1) dM_X(x)\right)\right\},$$

where τ_X is the trend term of the process.

If $Y \geqq 0$ is independent of $\{X(t)\}$, then it is well known that the superposition $X(Y)$ has characteristic function

$$f_{X(Y)}(u) = \int_{-\infty}^{\infty} f_{X(t)}(u) dF_Y(t).$$

Applying the Helly-Bray Theorem, we see that if $\langle Y_n \rangle$ and Y are independent of $\{X(t)\}$ and $Y_n \xrightarrow{\mathcal{L}} Y$, then $X(Y_n) \xrightarrow{\mathcal{L}} X(Y)$. The Lebesgue Dominated Convergence Theorem implies that if Y is independent of the differential processes $\{X_n(t)\}$ and $X_n(t) \xrightarrow{\mathcal{L}} X(t)$, $\forall t$, then $X_n(Y) \xrightarrow{\mathcal{L}} X(Y)$.

2. Superpositions of processes.

THEOREM 1. *Let $\{X(t)/t \in [0, \infty)\}$ be a differential process and let Y be a nonnegative random variable that is independent of the process. The distribution function of the superposition $X(Y)$ is given by*

$$\text{(A)} \qquad \begin{aligned} F_{X(Y)}(x) &= P[Y = 0] + \int_0^{\infty} F_{X(t)}(x) dF_Y(t), & x &\geqq 0, \\ &= \int_0^{\infty} F_{X(t)}(x) dF_Y(t), & x &< 0, \end{aligned}$$

at points of continuity.

The proof is sketched for $x \geqq 0$ fixed. We introduce auxiliary processes $\{Z_n(t)\}$ to avoid the complication caused by the fact that the stochastic distribution function $F_{X(t)}(x)$ may not be continuous as a function of t. For each positive integer n let $\{Z_n(t)\}$ be a differential process, independent of $\{X(t)\}$ and Y, such that $Z_n(t)$ is a nonpositive random variable with characteristic function

$$f_{Z_n(t)}(u) = \exp\left\{\frac{t}{n}\int_{-\infty}^{0}(e^{iux}-1)d_x(1/|x|^{1/2})\right\}.$$

(Such processes can be generated via the first exit times of a Wiener process.) Since the spectral function assigns infinite mass to $(-\infty, 0)$, we know that $Z_n(t)$ has continuous distribution. The Central Limit Theorem and the nonpositiveness guarantee that $F_{Z_n(t)}(\cdot)$ converges *everywhere* to the distribution with all mass at the origin as n tends to ∞. Now consider the infinitely divisible processes $\{X_n(t) = X(t) + Z_n(t)\}$. By the Lebesgue Dominated Convergence Theorem

$$\begin{aligned} F_{X_n(t)}(x) &= \int_{-\infty}^{\infty} F_{Z_n(t)}(x-y)dF_{X(t)}(y) \\ &\to \int_{-\infty}^{x} dF_{X(t)}(y) = F_{X(t)}(x), \qquad \forall x. \end{aligned}$$

Moreover, $\{X_n(t)\}$ is continuous in law and each random variable $X_n(t)$ has continuous distribution. Thus $F_{X_n(t)}(x)$ is continuous as a function of t.

Now set

$$Y_m = 0\cdot I_{[Y=0]} + \sum_{k=1}^{\infty}\frac{k}{2^m}\cdot I_{[(k-1)/2^m<Y\leq k/2^m]}.$$

Then

$$\begin{aligned} F_{X_n(Y_m)}(x) &= P[X_n(Y_m) \leqq x] \\ &= P[Y=0] + \sum_{k=1}^{\infty} P[X_n(k/2^m) \leqq x]\cdot P[Y_m = k/2^m] \\ &= P[Y=0] + \sum_{k=1}^{\infty} F_{X_n(k/2^m)}(x)\cdot P[Y_m = k/2^m] \\ &= P[Y=0] + \int_0^{\infty} F_{X_n(t)}(x)dF_{Y_m}(t). \end{aligned}$$

Now $Y_m \downarrow Y$ so $F_{Y_m} \xrightarrow{c} F_Y$. Thus the Helly-Bray Theorem yields

$$F_{X_n(Y)}(x) = P[Y = 0] + \int_0^\infty F_{X_n(t)}(x) dF_Y(t).$$

An application of the Lebesgue Dominated Convergence Theorem completes our proof.

Applying the Lebesgue Dominated Convergence Theorem to (A), we obtain

COROLLARY 1A. *Let $\{X(t)\}$ be a differential process with continuous distributions and let Y be a nonnegative random variable that is independent of the process with $P[Y=0]=\gamma$. Then the probability distribution function of the superposition is continuous except for a jump of size γ at the origin.*

3. Superpositions of processes with discontinuous distributions.

THEOREM 2. *Let $\{X(t)\}$ be any differential process with discontinuous distributions and zero trend term. Then no superposition can have continuous distribution.*

PROOF. We shall show that such a superposition must have a discontinuity at the origin. (A) clearly implies such a discontinuity if $P[Y=0]>0$ so, without loss of generality, we assume $Y>0$. If the process has discontinuous distributions, then we know that $\int_{-\infty}^{\infty} dM_X(x) < \infty$. Let $\mu = \int_{-\infty}^{0} dM_X(x)$ and $\lambda = \int_0^\infty dM_X(x)$. We may assume that $\mu+\lambda>0$ since the claimed result is trivial if $X(t)\equiv 0$, $\forall\, t$. If we define a distribution function G via

$$\begin{aligned} G(x) &= M_X(x)/\mu + \lambda, & x &< 0, \\ &= \mu/\mu + \lambda, & x &= 0, \\ &= (\mu + \lambda + M_X(x))/\mu + \lambda, & x &> 0, \end{aligned}$$

then $F_{X(t)}(\cdot)$ is the distribution function of the random sum $Z = X_1 + \cdots + X_Y$, where $X_1, X_2, \cdots$ are independent with common distribution G and $\mathcal{L}(Y) = \mathcal{P}(t(\mu+\lambda))$. This follows from Theorem 6 of §6.1 in [4]. Indeed if $g(u)$ is the characteristic function of X_1, then

$$\begin{aligned} f_Z(u) &= \exp\{t(\mu + \lambda)[g(u) - 1]\} \\ &= \exp\left\{t(\mu + \lambda) \int_{-\infty}^{\infty} (e^{iux} - 1) dG(x)\right\} \\ &= \exp\left\{t \int_{-\infty}^{\infty} (e^{iux} - 1) dM_X(x)\right\}. \end{aligned}$$

It is now clear that $F_{X(t)}(\cdot)$ has a jump at the origin of magnitude at least $e^{-t(\mu+\lambda)}$. Using (A) and the Lebesgue Dominated Convergence Theorem, we see that

$$F_{X(Y)}(0) - F_{X(Y)}(0-) \geqq \int_0^\infty e^{-t(\mu+\lambda)} dF_Y(t) > 0$$

and the superposition has a discontinuity at the origin.

The preceding result says that a differential process with discontinuous distributions and no trend or drift is in some sense "bad" and the discontinuities cannot be smoothed out by means of a superposition. The presence of a trend term, however, insures that some superpositions must be continuous. In particular, suppose that Y is a *random epoch*; i.e., a nonnegative infinitely divisible random variable with spectral function satisfying $\int_{+0}^{+1} x dM_Y(x) < \infty$. The question of whether or not the superposition has continuous distribution can be answered by

THEOREM 3. *Let $\{X(t)\}$ be a differential process with discontinuous distributions and nonzero trend term. Let Y be any independent random epoch. Then the superposition $X(Y)$ has continuous distribution if and only if Y has continuous distribution.*

PROOF. V. M. Zolotarev in [5] and the author in [3] have shown that the superposition is an infinitely divisible random variable with no gaussian component and Lévy spectral function

$$M^*(x) = \tau_Y M_X(x) + \int_0^\infty F^*_{X(t)}(x) dM_Y(t),$$

where the stochastic distribution function is defined via

$$\begin{aligned} F^*_{X(t)}(x) &= F_{X(t)}(x), & x < 0, \\ &= F_{X(t)}(x) - 1, & x > 0. \end{aligned}$$

We note that the superposition has continuous distribution if and only if $\int_{-\infty}^{\infty} dM^*(x) = \infty$. It is clear that this cannot happen if $\int_0^\infty dM_Y(t) < \infty$ and the necessity of Y having continuous distribution is easily established.

Now suppose that Y has continuous distribution and that the differential process has positive trend. We shall show that the spectral function of the superposition must assign infinite mass to the positive half-axis. Using the argument and notation of Theorem 2, we see that $F_{X(t)}(\cdot)$ takes a jump of magnitude at least $e^{-t(\mu+\lambda)}$ at $t\tau_X$. Thus for $x>0$, the superposition has spectral function

$$\begin{aligned} M^*(x) &= \int_0^\infty F^*_{X(t)}(x)dM_Y(t) \\ &\leqq \int_{x/\tau}^\infty F^*_{X(t)}(x)dM_Y(t) \\ &\leqq -\int_{x/\tau_X}^\infty e^{-t(\mu+\lambda)}dM_Y(t). \end{aligned}$$

The Lebesgue Monotone Convergence Theorem now implies that $M^*(0+)=-\infty$. and the superposition has continuous distribution. A similar argument for a differential process with negative trend completes the proof.

It is not clear that Theorem 3 would be valid if Y were not a random epoch. The technique of the proof obviously cannot be extended.

References

1. J. R. Blum and M. Rosenblatt, *On the structure of infinitely divisible distributions*, Pacific J. Math. **9** (1959), 1–7. MR **21** #4465.

2. P. Hartman and A. Wintner, *On the infinitesimal generators of integral convolutions*, Amer. J. Math **64** (1942), 273–298. MR **4**, 18.

3. B. W. Huff, *The strict subordination of differential processes*, Sankyā Ser. A **31** (1969), 403–412.

4. H. G. Tucker, *A graduate course in probability*, Probability and Math. Statist., vol. 2, Academic Press, New York, 1967. MR **36** #4593.

5. V. M. Zolotarev, *Distribution of the superposition of infinitely divisible processes*. Teor. Verojatnost. i Primenen. **3** (1958), 197–200 = Theor. Probability Appl. **3** (1958), 185–188. MR **23** #A683.

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