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ON POINTWISE PERIODIC TRANSFORMATION GROUPS

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ABSTRACT. Let X be a connected and metrizable manifold without boundary, and (X, T) a transformation group. We prove that if T is countable and pointwise periodic then T is periodic. This is a generalization of a result of Montgomery, which says that if h is a pointwise periodic homeomorphism of X onto itself then h is periodic.

We give in the following a generalization of a theorem of Montgomery (see [2]). For notation and terminology in the following see [1]. Throughout we denote by X a metric space with metric d, and by (X, T) a transformation group. We also assume that T is countable. We say that $T=EA$ is a *decomposition* of T for $x\in X$ if E, A are subsets of T, $xE=x$ and A is compact. T is *periodic at* $x\in X$ if there is a decomposition of T for x; T is *pointwise periodic* if it is periodic at each point of X; $T=EA$ is a decomposition of T for $Y\subseteq X$ if it is a decomposition of T for each y in Y; T is *periodic* if there is a decomposition of T for X. The main result we wish to prove is the following.

THEOREM A. *Suppose X is a connected manifold without boundary. If T is pointwise periodic then it is periodic.*

1. For the purposes of this article assume further that T is pointwise periodic. If $T=EA$ is a decomposition of T for some $x\in X$, then $xT=xEA=xA$ is a compact and countable subset of X. Hence some point of xA is an isolated point of xA, and since T is transitive on xT, each point of xA is an isolated point of xA. Hence xA, being compact, is finite. Let Z denote the set of all positive integers with the order topology, and define a function $\phi: X\to Z$ by setting, for any $x\in X$, $x\phi$ to be the cardinality of xT. We agree to denote a δ-neighbourhood of $x\in X$ by $U(x, \delta)$.

LEMMA 1. *Let $x\in X$ and $T=EA$ be a decomposition of T for x. Given any $\epsilon>0$, there exists a $\delta>0$, such that, for any $f\in A$, $U(x, \delta)f \subseteq U(xf, \epsilon)$ and for any $f, g\in A$, if $xf\neq xg$ then $U(x, \delta)f\cap U(x, \delta)g=\emptyset$.*

PROOF. Let $xT=xA=\{x_i : 1\leqq i\leqq n\}$, and 3η be the least of all the

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real numbers $\{d(x_i, x_j): 1 \leqq i, j \leqq n, i \neq j\}$ and ϵ. Since A is a compact subset of T, it is equicontinuous at x. Hence there exists a $\delta > 0$, such that, for any $f \in A$, $U(x, \delta)f \subseteq U(xf, \eta) \subseteq U(xf, \epsilon)$. If $x_i = xf \neq xg = x_j$, for some f, g in A, then $d(x_i, x_j) \geqq 3\eta$ and therefore $U(x, \delta)f \cap U(x, \delta)g = \emptyset$. This completes the proof.

THEOREM 1. *ϕ is lower semicontinuous on X.*

PROOF. Let $x \in X$, $x\phi = n \in Z$, and $xT = \{x_i: 1 \leqq i \leqq n\}$. Let $\epsilon = \min\{d(x_i, x_j): 1 \leqq i, j \leqq n, i \neq j\}$, and $\delta > 0$ be as in Lemma 1. Then, since for any $y \in U(x, \delta)$, each $U(x_i, \epsilon)$, $1 \leqq i \leqq n$, contains at least one element of yT, $y\phi \geqq n$. This completes the proof.

THEOREM 2. *Suppose X is locally connected at $x \in X$. If ϕ is continuous at x, then T is equicontinuous at x.*

PROOF. Let $\epsilon > 0$ be given. Let $T = EA$ be a decomposition of T for x, $xT = \{x_i: 1 \leqq i \leqq n\}$, and $A_i = \{f \in A: xf = x_i\}$, $1 \leqq i \leqq n$. Since ϕ is continuous at x there exists a $\delta_1 > 0$, such that, $U(x, \delta_1)\phi = n$. Let $\delta_2 > 0$ be as in Lemma 1 with respect to $\epsilon/2$. Let V be a connected open set $\subseteq U(x, \delta)$, and containing x, where $\delta = \min(\delta_1, \delta_2)$, and $W_i = VA_i$, $1 \leqq i \leqq n$. Then $\{W_i: 1 \leqq i \leqq n\}$ is a disjoint family of open sets, and the diameter of each W_i is less than ϵ (see Lemma 1).

Let $y \in V$, and $T = FB$ be a decomposition of T for y. Then $yA \subseteq yT = yB$. Since $yA_i \subseteq W_i$, $1 \leqq i \leqq n$, the cardinality of $yA \geqq n$. But since n is finite, and $y\phi = n$, for $\delta \leqq \delta_1$, we must have the cardinality of $yA = n$. Hence $yA = yB$. Thus $yT = yB \subseteq \bigcup\{W_i: 1 \leqq i \leqq n\}$. Since $y \in V$ above was arbitrary, $VT \subseteq \bigcup\{W_i: 1 \leqq i \leqq n\}$.

Let $f \in T$ be arbitrary. Then $f = gh_i$, where $g \in E$ and $h_i \in A_i$ for some i. Since Vf is connected, and $Vf \cap W_i \neq \emptyset$ and $\{W_i: 1 \leqq i \leqq n\}$ is a disjoint family of open sets covering Vf, it follows that $Vf \subseteq W_i$. Since the diameter of W_i is less than ϵ, for any $y \in V$, $d(yf, xf) < \epsilon$. Since V is open, it follows that T is equicontinuous at x. This completes the proof.

LEMMA 2. *Let $x \in X$ be a point of continuity of ϕ and of equicontinuity of T. Then there exists a $\delta > 0$, such that, for any two points of $U(x, \delta)$ the decomposition of T for one is a decomposition of T for the other.*

PROOF. Let $T = EA$ be a decomposition of T for x, $xT = \{x_i: 1 \leqq i \leqq n\}$, and $A_i = \{f \in A: xf = x_i\}$, $1 \leqq i \leqq n$. Let 2ϵ be the smallest of all $d(x_i, x_j)$, $1 \leqq i, j \leqq n$, $i \neq j$. Then from the continuity of ϕ and the equicontinuity of T at x, there exists a $\delta > 0$, such that, $U(x, \delta)\phi = n$, and for any $f \in T$, $U(x, \delta)f \subseteq U(xf, \epsilon)$. Let $U(x, \delta) = U$.

Let $f\in E$ and $y\in U$. If $yf\neq y$, then $U(x_i, \epsilon)$, where $x_i=x$, contains two distinct elements y and yf. But each $U(x_i, \epsilon)$, $1\leqq i\leqq n$, contains at least one element of yT, and any two of them are mutually disjoint, implying that $y\phi>n$. This is a contradiction since $y\in U$. Hence $yf=y$. Since $y\in U$ and $f\in E$ were arbitrary $T=EA$ is a decomposition for U.

Let $W_i=UA_i$, $1\leqq i\leqq n$. From the choice of ϵ and δ above, $W_i\cap W_j=\varnothing$ if $i\neq j$, $1\leqq i, j\leqq n$. Let $f\in T$. Then from the decomposition $T=EA$, $f=hk$, where $h\in E$ and $k\in A$. But from the last paragraph above $h\in E$ implies that h is the identity on U. Hence $f=k$ on U. Now if for some $y\in U$, $yf=y$, then $Uk\cap U\neq\varnothing$. Since $U\subseteq U(x, \epsilon)$ and $Uk\subseteq U(xk, \epsilon)$, it follows that $Uk\subseteq U(x, 2\epsilon)$.

Hence from the choice of ϵ, $xk=x$, and consequently $xf=x$. This implies that any decomposition of T for $y\in U$ is also a decomposition for x.

The above two results imply the statement of the lemma and complete the proof.

THEOREM 3. *Suppose R is a connected open subset of X on which ϕ is continuous and T is equicontinuous. Then any decomposition of T for any point of R is a decomposition for R.*

PROOF. For each $x\in R$ there exists an $\eta(x)$ as in Lemma 2. Consider the open covering $\{U(x, \eta(x)):x\in R\}$ of R. Let y be any given point of R, and $T=EA$ be a decomposition of T for y. Since R is connected, given any $z\in R$, there exists a finite set, $y=x_1, x_2, \cdots, x_n=z$ such that

$$U(x_i, \eta(x_i)) \cap U(x_{i+1}, \eta(x_{i+1})) \neq \varnothing, \qquad 1 \leqq i \leqq n-1.$$

Hence from Lemma 2 it follows, inductively, that $T=EA$ is a decomposition of T for z. Since $z\in R$ was arbitrary the decomposition of T for y is a decomposition for R. This completes the proof.

The following is a consequence of Theorems 2 and 3.

THEOREM 4. *Suppose X is locally connected, and M is the set of all points of continuity of ϕ. Then the decomposition of T for any point of a component R of M is a decomposition for R. Indeed T is periodic on each component of M.*

2. **Proof of Theorem** A. Let $M=\{x\in X:\phi$ is continuous at $x\}$. Then M is everywhere dense in X, and since Z is discrete, it is also open in X. Let R be a component of M. Then from Theorem 4 there is a decomposition $T=EA$ of T for R. Let $f\in E$. Since T is periodic at each point $x\in X$, it is easy to see that f is also periodic at each

point of X. Hence, from Montgomery's theorem [2], f is periodic on X. But then Newman's Theorem 1 [3] implies, since f is the identity on a connected open set R in X, that f is identity on X. Hence $T=EA$ is a decomposition of T for X. This completes the proof.

REMARK. Suppose T is not necessarily countable as assumed above. Define T to be *discretely periodic* at $x\in X$ if there exist subsets E, A of T, such that, $T=EA$, $xE=x$ and A is finite. We say that T is discretely periodic if there exist subsets E, A of T, such that, $T=EA$, A is finite and $xE=x$ for each $x\in X$. Then each result in §1 above can be proved for T which is discretely periodic at each $x\in X$, and consequently the following:

THEOREM B. *If X is a connected manifold without boundary, and T is discretely periodic at each point of X, then T is discretely periodic.*

REFERENCES

1. W. E. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R. I., 1955. MR **17**, 650.

2. D. Montgomery, *Pointwise periodic homeomorphisms*, Amer. J. Math. **59** (1937), 118–120.

3. M. H. A. Newman, *A theorem on periodic transformations of spaces*, Quart. J. Math. **2** (1931), 1–8.

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