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ON CONSTRUCTING NEARLY DECOMPOSABLE MATRICES

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ABSTRACT. In this paper we consider the construction of nearly decomposable matrices from nearly decomposable matrices of smaller dimension. In particular, if $A_1, A_2, \cdots, A_s$ are nearly decomposable matrices we consider the problem of finding matrices $E_1, E_2, \cdots, E_s$ each with exactly one nonzero entry so that

$$\begin{bmatrix} A_1 & 0 & \cdots 0 & E_1 \\ E_2 & A_2 & \cdots 0 & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots E_s & A_s \end{bmatrix}$$

is nearly decomposable.

Introduction and notation. In [6] Sinkhorn and Knopp introduced the notion of nearly decomposable matrices. These matrices were used to solve a conjecture of Marshall Hall in [5], and to find lower bounds for the permanent function on certain classes of (0, 1)-matrices in [4]. The structure of these matrices is given below.

THEOREM I. *Suppose A is an $n\times n$ nonnegative matrix, $n>1$. If A is nearly decomposable then there exist permutation matrices P and Q and an integer $s>1$ so that*

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots 0 & E_1 \\ E_2 & A_2 & \cdots 0 & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots E_s & A_s \end{bmatrix}$$

where each A_k is nearly decomposable and $E_1, E_2, \cdots, E_s$ have exactly one nonzero entry [6].

In [2] it is shown that $A_1, A_2, \cdots, A_{s-1}$ may be assumed to be of dimension one.

In this paper we consider ways in which nearly decomposable matrices may be constructed from nearly decomposable matrices of lower dimension. In particular, if $A_1, A_2, \cdots, A_s$ are nearly decomposable matrices we wish to find where positive numbers may be placed in the E_k's so that

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$$\begin{pmatrix} A_1 & 0 & \cdots 0 & E_1 \\ E_2 & A_2 & \cdots 0 & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots E_s & A_s \end{pmatrix}$$

is nearly decomposable.

Of course, if we arbitrarily place a positive number in each E_k then the resulting matrix is fully indecomposable by the following theorem.

THEOREM II. *Suppose A is an $n \times n$ nonnegative matrix of the form*

$$\begin{pmatrix} A_1 & 0 & \cdots 0 & E_1 \\ E_2 & A_2 & \cdots 0 & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots E_s & A_s \end{pmatrix}$$

where $s > 1$ and for $k = 1, 2, \cdots, s$; A_k is fully indecomposable and E_k has exactly one positive entry. Then A is fully indecomposable [6, *p.* 69].

This does not imply however that the resulting matrix is nearly decomposable. In fact, examples can easily be found where this is not the case.

We include the following definitions and notation.

A nonnegative n-square matrix A, $n > 1$, is partly decomposable if there exist permutation matrices P and Q so that $PAQ = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$ where X and Z are square. Otherwise A is fully indecomposable. By convention a 1×1 matrix is fully indecomposable if and only if its single entry is positive. Let E_{ij} be an n-square (0, 1)-matrix having a one in the ijth position and 0's elsewhere. If $A = (a_{ij})$ is fully indecomposable and for each $a_{ij} \neq 0$ we have $A - a_{ij}E_{ij}$ partly decomposable, then we say that A is nearly decomposable.

If σ is a permutation of $1, 2, \cdots, n$ and $a_{1\,\sigma(1)}, \cdots, a_{n\,\sigma(n)}$ are all positive entries of A then we call $a_{1\,\sigma(1)}, \cdots, a_{n\,\sigma(n)}$ a positive diagonal of A. Each $a_{k\,\sigma(k)}$ is said to lie on the positive diagonal. If each $a_{ij} \neq 0$ in A lies on some positive diagonal of A then we say that A has total support. By using the Frobenius-Konig Theorem [3, p. 97] it is easily shown that fully indecomposable implies total support.

In order to simplify much of the following we denote by $A = (a_{ij})$ an $n \times n$ nearly decomposable (0, 1)-matrix in the (A_k, E_k) form of Theorem I. Further suppose that the one in E_k is in the (i_k, j_k) position of A.

Results. In order to construct nearly decomposable matrices from nearly decomposable matrices of lower dimension we introduce the following definition.

DEFINITION. Suppose $(p_1, q_1), \cdots, (p_r, q_r)$ are 0 positions in A. Let B denote the $n\times n$ (0, 1)-matrix with 1's only in the $(p_1, q_1), \cdots, (p_r, q_r)$ positions. Suppose $A+B$ has the property that if any 1 in A is replaced by a 0 yielding $\overline{A}$ we have that $\overline{A}+B$ is partly decomposable. We then call $\{(p_1, q_1), \cdots, (p_r, q_r)\}$ a tie set for A. A position (p, q) for which $\{(p, q)\}$ is a tie set is called a tie point for A.

EXAMPLE.

(1) The tie set is empty.

$\begin{pmatrix} 11 \\ 11 \end{pmatrix}$ The tie set is empty.

$\begin{pmatrix} 110 \\ 011 \\ 101 \end{pmatrix}$ Three nonempty tie sets. $\{(1, 3)\}$ $\{(2, 1)\}$ $\{(3, 2)\}$

It may also be noted that a subset of a tie set is itself a tie set.

We now show how to find tie sets. For this we need the following lemma.

LEMMA. *Suppose B is an $n\times n$ fully indecomposable* (0, 1)-*matrix, $n>1$. If some* 1 *in B is replaced by a* 0, *the resulting matrix $\overline{B}$ is partly decomposable if and only if $\overline{B}$ does not have total support.*

PROOF. If $\overline{B}$ is partly decomposable then there exist permutation matrices P and Q so that

$$P\overline{B}Q = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

where X and Z are square. Since B was fully indecomposable, Y is not 0. Hence $\overline{B}$ does not have total support.

Conversely if $\overline{B}$ does not have total support, then it is partly decomposable since any fully indecomposable matrix has total support.

THEOREM 1. *If A_k is not 1×1 then the (i_k, j_{k+1}) position is a* 0 *position of A; $k, k+1$* mod s. *$\{(i_k, j_{k+1}) \mid A_k$ is not $1\times 1\}$ is a tie set for A. Further the position (i_k, j_{k+1}) is a tie point for A_k for each A_k not 1×1.*

PROOF. Suppose there is a 1 in the (i_k, j_{k+1}) position of A, A_k not 1×1. Replace this 1 in A by 0 calling this matrix $\overline{A}$ and having $(\overline{A}_l, E_l)$ form:

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 & \cdots 0 & E_1 \\ E_2 & \bar{A}_2 & \cdots 0 & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots \bar{A}_{s-1} & 0 \\ 0 & 0 & \cdots E_s & \bar{A}_s \end{bmatrix}.$$

Now $\bar{A}_k$ and $\bar{A}$ do not have total support since A_k and A are nearly decomposable. Since A_k has at least two 1's in each row and column and has total support there exists at least one positive diagonal in $\bar{A}_k$. Therefore if $l \neq k$, then every 1 in $\bar{A}_l$ is on a positive diagonal of $\bar{A}$. The 1's in the E_p's ($p=1, 2, \cdots, s$) are also still on a positive diagonal of $\bar{A}$. Pick some 1 in $\bar{A}_k$. If it is on a positive diagonal in $\bar{A}_k$ then it is on a positive diagonal in the $\bar{A}_l$'s of $\bar{A}$. If it is not on a positive diagonal of $\bar{A}_k$, then it is on a positive diagonal of A_k, and hence on a positive diagonal with the 1's in the E_p's in $\bar{A}$. But now $\bar{A}$ has total support. This gives us a contradiction. Hence it follows that there must have been a 0 in the (i_k, j_{k+1}) position of A.

To show $\{(i_k, j_{k+1}) \mid A_k \text{ not } 1\times 1\}$ is a tie set for A we proceed as follows. Denote by B the $n\times n$ (0, 1)-matrix with 1's in the (i_k, j_{k+1}) positions and 0's elsewhere, A_k not 1×1. Hence

$$B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_s \end{bmatrix}$$

where each B_k is square and of the same dimension as A_k. Now $A+B$ is an $n\times n$ (0, 1)-matrix. Replace some 1 in A by 0 calling this matrix $\bar{A}$ having $(\bar{A}_l, \bar{E}_l)$ form. We now show $\bar{A}+B$ is partly decomposable.

If the 1 replaced by 0 in A was in some E_k, or in some A_k which was 1×1, then $\bar{A}+B$ is clearly partly decomposable. Therefore we suppose the 1 replaced by 0 in A was in A_r for some r so that A_r is not 1×1. For this, suppose $\bar{A}+B$ is fully indecomposable and hence has total support. Now $\bar{A}_r$ still has at least one positive diagonal. Hence each 1 in $\bar{A}_k$, $k\neq r$, is on a positive diagonal in $\bar{A}$. Since $\bar{A}+B$ has total support, the 1's in the E_k's are on a positive diagonal in $\bar{A}$. Pick any 1 in $\bar{A}_r$. If it is on a positive diagonal with the 1 in the (i_r, j_{r+1}) position in $\bar{A}_r+B_r$, then it is on a positive diagonal with the 1's in the E_k's in $\bar{A}$. If it is not on a positive diagonal with the 1 in the (i_r, j_{r+1})

position in $\bar{A}_r+B_r$, then it is on a positive diagonal in the $\bar{A}_k$'s of $\bar{A}$. Hence we have shown that $\bar{A}$ has total support which is a contradiction to A being nearly decomposable. Therefore $\{(i_k, j_{k+1}) \mid A_k$ is not $1\times 1\}$ is a tie set for A.

As above, it can be shown that if $\bar{A}_r+B_r$ is fully indecomposable, then so is $\bar{A}$. Hence we see that $\bar{A}_r+B_r$ is partly decomposable and hence (i_r, j_{r+1}) is a tie point for A_r.

COROLLARY 1. *Suppose A_1 in A is not 1×1. If a 1 in A_1 is replaced by 0, there results a matrix $\bar{A}$, and there exist permutation matrices P and Q so that*

$$P\bar{A}Q = \begin{pmatrix} \begin{array}{cc|} X & 0 \\ Y & Z \\ \hline \end{array} & 0 \cdots 0 & 0 \\ & 0 \cdots 0 & F \\ E & A_2 \cdots 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots A_{s-1} & 0 \\ 0 & 0 \cdots E_s & A_s \end{pmatrix}$$

or

$$P\bar{A}Q = \begin{pmatrix} \begin{array}{cc|} X & 0 \\ Y & Z \\ \hline \end{array} & 0 \cdots 0 & E \\ F \quad 0 & 0 \cdots 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 \quad 0 & 0 \cdots A_{s-1} & 0 \\ 0 \quad 0 & 0 \cdots E_s & A_s \end{pmatrix}$$

where X and Z are square, E and F have exactly one entry equal to 1.

PROOF. (i_1, j_2) is a tie point for A_1. This determines the intersection of row i_1 and column j_2.

COROLLARY 2. *No A_k of A is* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

COROLLARY 3. *A has a nonempty tie set if and only if A is not* 1×1 *or* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

PROOF. If each A_k of A $(k=1, \cdots, s)$ is 1×1 then any 0 position will do. If some A_k of A is not 1×1, then it cannot be 2×2 by the preceding corollary, hence (i_k, j_{k+1}) will be a tie point for A.

We now show how to use tie sets in order to construct nearly decomposable matrices.

THEOREM 2. *If $A_1, A_2, \cdots, A_s$ are nearly decomposable matrices, no A_k being $\binom{1\ 1}{1\ 1}$ then there exists some E_k's so that*

$$\begin{pmatrix} A_1 & 0 & 0\cdots 0 & E_1 \\ E_2 & A_1 & 0\cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots A_{s-1} & 0 \\ 0 & 0 & 0\cdots E_s & A_s \end{pmatrix}$$

is nearly decomposable.

PROOF. Each A_k, A_k not 1×1, has a tie set. Take one point out of each tie set, say (i_k, j_k). If A_k is 1×1 then set $(i_k, j_k)=(1, 1)$. Now if A_k is $n_k\times n_k$ then consider the matrix A, $\sum n_k\times\sum n_k$ in (A_k, E_k) form by putting the A_k's along the main diagonal and the 1 in the E_k so that it is in row i_k of A_k and column j_{k-1} of A_{k-1}; k, $k-1$ mod s. We now show that A is nearly decomposable.

By Theorem II, A is fully indecomposable. To show A is nearly decomposable we proceed as follows. If a 1 in some E_k or some A_k where A_k is 1×1, is replaced by a 0, yielding $\overline{A}$, then $\overline{A}$ is clearly partly decomposable. Further if a 1 in some A_k, A_k not 1×1, is replaced by a 0 yielding $\overline{A}$, then since (i_k, j_k) is a tie point for A_k there exist permutation matrices P and Q so that $PA_kQ=\left\{\begin{smallmatrix} X & 0 \\ Y & Z\end{smallmatrix}\right\}$ where X and Z are square and the 0 in the (i_k, j_k) position of A_k is now in X, Y or Z. Hence, since the tie point (i_k, j_k) determines the intersection of row i_k in E_k and column j_k in E_{k+1} we see that $\overline{A}$ is partly decomposable, i.e., if I_l denotes the $n_l\times n_l$ identity matrix then

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \begin{bmatrix} P & 0 \\ 0 & I_{k+1} \end{bmatrix} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \begin{bmatrix} E_k & A_k \\ 0 & E_{k+1} \end{bmatrix} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

$$\cdot \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \begin{bmatrix} Q & 0 \\ 0 & I_{k+1} \end{bmatrix} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \begin{bmatrix} 0 & \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \\ F & \\ 0 & E \end{bmatrix} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

or

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & \begin{bmatrix} E & \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \\ 0 & F \quad 0 \end{bmatrix} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

COROLLARY 1. *If $A_1, A_2, \cdots, A_s$ are nearly decomposable, then there exist some E_k's so that*

$$\begin{pmatrix} A_1 & 0 & \cdots 0 & E_1 \\ E_2 & A_2 & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots A_{s-1} & 0 \\ 0 & 0 & \cdots E_s & A_s \end{pmatrix}$$

is nearly decomposable if and only if no A_k is $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

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