PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, No. 2, February 1971

COMPLEMENTS TO SOLVABLE HALL SUBGROUPS[1]

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ABSTRACT. A Hall subgroup H of a finite group G is a subgroup whose order is relatively prime to its index. We show that if H is solvable and if the way prime power elements of H are conjugate in G is restricted, then G has a quotient isomorphic to H.

Suppose H is a Hall subgroup of G. Let π be the set of prime divisors of the order of H. A number is π if all its prime divisors are in π and π' if none of them are. The index $(G:H)$ is π'. If H has a normal complement in G, then any two elements of H conjugate in G are conjugate by an element of H; i.e. H is c-closed. Taking H to be the permutation group on four letters and G to be the one on five letters shows that a c-closed solvable Hall subgroup need not have a normal complement. However if H has a normal complement we also know that for any subgroup D of H the centralizer in H of D is a Hall subgroup of the centralizer of D. Thus we have

(1) If H has a normal complement, then H is c-closed and $(C(D):C_H(D))$ is π'.

If H is a solvable Hall π subgroup satisfying (1), then by Proposition 1, H has a normal complement. Propositions 2 and 3 give other sufficient conditions. From now on we assume H is a solvable Hall π subgroup of G.

PROPOSITION 1. *If for every $x\in G$ and prime power element $h\in H$ such that $h^x=x^{-1}hx\in H$ there exist elements $y_1 \cdots y_k$ and subgroups $B_1 \cdots B_k$ satisfying*

(a) $x=y_1 \cdots y_k$,

(b) $y_i\in B_iH$, *the set of products of elements from B_i and H,*

(c) $B_1\subset C(h)$, $B_i\subset C(h^{y_1\cdots y_{i-1}})$ *for* $2\leqq i\leqq k$,

(d) $(B_i:H\cap B_i)$ *is* π',

then H has a normal complement.

PROOF. By Corollary 1 to Theorem 9 and the corollary to Theorem 10* of [2, §5] or by a transfer argument we can find N normal in G such that $HN=G$, $N\neq G$. Let $M=H\cap N$, $C_i=B_i\cap N$. We claim that

Received by the editors April 13, 1970.

AMS 1970 *subject classifications*. Primary 20D20; Secondary 20D10.

Key words and phrases. Solvable Hall subgroup, normal complement, finite group, quotient group, c-closed, transfer, prime power, fusion.

[1] Contains part of the author's dissertation.

if $x \in N$, then (a) through (d) still hold if we substitute M for H and C_i for B_i. It is easy to see that (a), (c) and (d) are valid; we prove (b) by induction on k. Since $(G:N)$ is π, we have by (d) $B_i = C_i(B_i \cap H)$ and $B_i H = C_i H$. If $k=1$, $y_1 \in B_1 H \cap N$ implies $y_1 \in C_1 M$. If $k \geqq 2$, let $z = y_1 \cdots y_{k-1}$. By (b) and (c) $h^z \in H$, and so by induction $y_i \in C_i M \subset N$. As before $y_k = z^{-1}x \in B_k H \cap N$ implies $y_k \in C_k M$.

If we restrict k to be 1, then Proposition 1 and (1) give the following corollary:

COROLLARY 1. *H has a normal complement iff*
(a) *any two prime power elements of H which are conjugate in G are conjugate in H,*
(b) *for any p.p. element $h \in H$, $(C(h): C_H(h))$ is π'.*

We remark that if H is not a solvable Hall subgroup of G, but $(G:H)$ is π' and the other hypotheses of Proposition 1 hold, then the proof yields that the maximal solvable π quotients of G and H are isomorphic.

COROLLARY 2 (BRAUER AND SUZUKI). *H has a normal complement iff*
(a) *any two p.p. elements of H which are conjugate in G are conjugate in H,*
(b) *any subgroup E which is the product of a cyclic p group and a q group for p, $q \in \pi$ has a conjugate in H.*

PROOF. We show that the hypotheses imply that (b) of Corollary 1 holds. Let E be the direct product of the cyclic group generated by h and a Sylow q subgroup Q of $C(h)$ for any $q \in \pi$.

By (b), $E^x \subset H$. Hence $h^x \subset H$ and $h^x = h^k$ for some $k \in H$ by (a). Taking $y = xk^{-1}$ we have $Q^y \subset E^y \subset H \cap C(h^y) = C_H(h)$. The converse follows from the Schur-Zassenhaus Theorem.

Let H^* be the subgroup of H generated by all intersections $H \cap H^x$, $x \notin N(H)$. If H is normal in G set $H^* = H$. Note that $N(H) \subset N(H^*)$.

PROPOSITION 2. *Suppose $G = RSR$ for two subgroups R and S satisfying $N(H) \subset R \subset N(H^*)$, $H \subset S$. If H has a normal complement in R and in S, then H has one in G.*

PROOF. We apply Proposition 1.

Suppose h, $h^x \in H$. If $h \notin H^*$, then $x \in R$ and by (1) we can take $y_1 = x$, $B_1 = C_R(h)$. Otherwise $x = rst$, r, $t \in R$, $s \in S$ and h^r, $h^{rs} \in H$. In this case $y_1 = r$, $y_2 = s$, $y_3 = t$ and $B_1 = C_R(h)$, $B_2 = C_S(h^r)$, $B_3 = C_R(h^{rs})$.

Because Proposition 1 refers to conjugation of prime power elements, we can apply Alperin's results on fusion. From the definition of a conjugation family [1, Definitions 5.1, 5.3] we obtain:

PROPOSITION 3. *For each* $p \in \pi$ *pick a Sylow* p *subgroup* H_p *of* H *and a conjugation family for* H_p. *Let* $F = \{(D_i, T_i)\}$ *be the union of these families and the set* $\{(H_p, N(H_p))\}$. *If for each* $(D_i, T_i) \in F$ *and* $h \in D_i$ *there exists a subgroup* B_i *satisfying* $T_i \subset B_i H$, $B_i \subset C(h)$, $(B_i : H \cap B_i)$ *is* π', *then* H *has a normal complement.*

In particular for the conjugation family of [1, Theorem 5.1] we have the following result:

COROLLARY 1. *If for each* $p \in \pi$ *and* p *subgroup* $D \subset H_p$ *we have*

(a) $(C(D) : C_H(D))$ *is* π',

(b) $N(D) = C(D) N_H(D)$ *if* $C_{H_p}(D) \not\subset D$,

then H *has a normal complement.*

BIBLIOGRAPHY

1. J. L. Alperin, *Sylow intersections and fusion*, J. Algebra **6** (1967), 222–241. MR **35** #6748.

2. R. Brauer, *A characterization of the characters of groups of finite order*, Ann. of Math. (2) **57** (1953), 357–377. MR **14**, 844.

3. ———, *On quotient groups of finite groups*, Math. Z. **83** (1964), 72–84. MR **28** #3088.

4. M. Suzuki, *On the existence of a Hall normal subgroup*, J. Math. Soc. Japan **15** (1963), 387–391. MR **28** #2157.

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