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A NOTE ON THE LOCATION OF CRITICAL POINTS OF POLYNOMIALS

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ABSTRACT. Let $\mathcal{P}(a, 3)$ denote the set of cubic polynomials which have all of their zeros in $|z| \leqq 1$ and at least one zero at $z=a$ ($|a| \leqq 1$). In this paper we describe a minimal region $\mathfrak{D}(a, 3)$ with the property that every polynomial in $\mathcal{P}(a, 3)$ has at least one critical point in $\mathfrak{D}(a, 3)$. The location of the zeros of the logarithmic derivative of the function $(z-a)^m(z-z_1)^{m_1}(z-z_2)^{m_2}$ is also discussed.

1. Introduction. Let $p(z)$ be a polynomial of degree n ($\geqq 2$) having all its zeros in the closed disk $\gamma: |z| \leqq 1$. Ilieff has conjectured [1, p. 25] that if a is a zero of $p(z)$, then at least one critical point of $p(z)$ (i.e., zero of $p'(z)$) lies in the disk $|z-a| \leqq 1$. A more difficult problem related to this conjecture is stated in [2] as follows:

Let $\mathcal{P}(a, n)$ be the set of all nth degree polynomials which have all of their zeros in γ and at least one zero at the point $z=a$. Describe a region $\mathfrak{D}(a, n)$ such that (i) $\mathfrak{D}(a, n)$ contains at least one critical point of every $p(z) \in \mathcal{P}(a, n)$ and such that (ii) no proper subset of $\mathfrak{D}(a, n)$ has property (i).

It is the aim of the present note to describe sets $\mathfrak{D}(a, 3)$, $|a| \leqq 1$, and thereby improve the results of Schmeisser [3] and others [4], [5] concerning the location of critical points of cubic polynomials. We shall define only those sets $\mathfrak{D}(a, 3)$ for which $0 \leqq a \leqq 1$, since the remaining sets can then be obtained by rotation. Our main result is

THEOREM 1. *Let $p(z) = (z-a)(z-z_1)(z-z_2)$, where $0 \leqq a \leqq 1$, $|z_1| \leqq 1$, and $|z_2| \leqq 1$. Let $\Delta(a)$ denote the closed disk*

$$\Delta(a): |z - a/2| \leqq ((4 - a^2)/12)^{1/2},$$

and $C(a)$ denote its circumference. Then $p'(z)$ has at least one zero in the set $\mathfrak{D}(a, 3)$ defined by

$$\mathfrak{D}(a, 3) = \Delta(a) \backslash [C(a) \cap \{z: \operatorname{Im} z > 0\}], \qquad a > 0,$$
$$\mathfrak{D}(0, 3) = \Delta(0).$$

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Furthermore if $\mathfrak{D}(a, 3)$ is replaced by any proper subset of $\mathfrak{D}(a, 3)$, then the assertion is false.

The proof of Theorem 1 is given in §2. In §3 we consider the location of the zeros of the logarithmic derivative of the function $(z-a)^m(z-z_1)^{m_1}(z-z_2)^{m_2}$, and in §4 we mention an open problem that is suggested by our results.

2. **Proof of Theorem 1.** The proof is based upon the following simple lemma:

LEMMA 1. *If $f(\zeta)=b_2\zeta^2+b_1\zeta+b_0$ is nonzero in $|\zeta|<1$, then*

$$|b_0|^2 - |b_2|^2 \geqq |\bar{b}_1 b_2 - \bar{b}_0 b_1|. \tag{1}$$

PROOF. The case $b_2=0$ is trivial. The case $b_2\neq 0$ is at once reduced to $b_2=1$.

Let α, β be the roots of $\zeta^2+b_1\zeta+b_0=0$, $|\alpha|\geqq 1$, $|\beta|\geqq 1$. Then $(|\alpha|-1)(|\beta|-1)\geqq 0$, and so $|\alpha\beta|+1\geqq|\alpha|+|\beta|$. Multiplying both sides of the last inequality by $|\alpha\beta|-1=|b_0|-1$ $(\geqq 0)$ there follows

$$\begin{aligned} |b_0|^2 - 1 &\geqq |\alpha|^2|\beta| + |\alpha|\,|\beta|^2 - |\alpha| - |\beta| \\ &= (|\alpha|^2-1)|\beta| + (|\beta|^2-1)|\alpha| \\ &\geqq |(|\alpha|^2-1)\bar{\beta} + (|\beta|^2-1)\bar{\alpha}| = |\bar{b}_1 - \bar{b}_0 b_1|, \end{aligned}$$

which proves Lemma 1.

We can now prove

LEMMA 2. *Let $p(z)=(z-a)(z-z_1)(z-z_2)$, where $0<a\leqq 1$, $|z_1|=1$, and $|z_2|\leqq 1$. If $p'(z)$ has no zero inside $C(a)$, then $z_2=\bar{z}_1$ or $a=z_1=1$ or $a=z_2=1$.*

Furthermore, if $p_\alpha(z)=(z-a)(z-e^{i\alpha})(z-e^{-i\alpha})$ and

$$0 \leqq \alpha_1 \equiv \cos^{-1}\left(\frac{a+6A}{4}\right) \leqq \alpha \leqq \cos^{-1}\left(\frac{a-6A}{4}\right) \equiv \alpha_2 \leqq \pi,$$

where $A\equiv((4-a^2)/12)^{1/2}$, then $p_\alpha'(z)$ has a pair of conjugate zeros and as α varies from α_1 to α_2 these zeros describe $C(a)$. If, however, $0\leqq\alpha<\alpha_1$ or $\alpha_2<\alpha\leqq\pi$, then $p_\alpha'(z)$ has a zero inside $C(a)$.

PROOF. Let $z_1=\exp[i\theta_1]$ and $z_2=r\exp[i\theta_2]$ $(0\leqq r\leqq 1)$. To prove the first part of the lemma we note initially that since $p'(z)$ is nonzero inside $C(a)$, the polynomial

$$P(\zeta) \equiv p'(A\zeta + a/2) = 3A^2\zeta^2 + A(a - 2z_1 - 2z_2)\zeta + z_1z_2 - a^2/4$$

is nonzero in $|\zeta|<1$. Hence as a consequence of (1) we have

$$|b_0|^2 - |b_2|^2 \geqq |\mathrm{Re}(\bar{b}_1 b_2 - \bar{b}_0 b_1)|, \tag{2}$$

where b_k denotes the coefficient of ζ^k in the expansion of $P(\zeta)$. Since

$$\operatorname{Re}(\bar{b}_1 b_2 - \bar{b}_0 b_1) = A\left[a(1 - r\cos(\theta_1 + \theta_2)) - 2(1 - r^2)\cos\theta_1\right]$$

and $A \geqq a/2$, we deduce from (2) that

$$\begin{aligned} a^2(1 - r\cos(\theta_1 + \theta_2))/2 - (1 - r^2) \\ \geqq a^2(1 - r\cos(\theta_1 + \theta_2))/2 - a\cos\theta_1(1 - r^2), \end{aligned}$$

that is $(1-r^2)(a\cos\theta_1 - 1) \geqq 0$. Hence either $r=1$ or $1 = a = \cos\theta_1 = z_1$. If $r=1$, then (2) implies that

$$(a/2 - A)(1 - \cos(\theta_1 + \theta_2)) \geqq 0,$$

and so either $a/2 = A$, i.e., $a=1$, or $\cos(\theta_1+\theta_2)=1$, i.e., $z_2 = \bar{z}_1$. Finally, if $r=1$ and $a=1$, we have equality in (2) and it follows from (1) that

$$\operatorname{Im}(\bar{b}_1 b_2 - \bar{b}_0 b_1) = \sin(\theta_1 + \theta_2) - \sin\theta_1 - \sin\theta_2 = 0.$$

This last equation implies that $\theta_1 \equiv 0$ or $\theta_2 \equiv 0$ or $\theta_1 + \theta_2 \equiv 0 \pmod{2\pi}$, i.e., $z_1 = 1$ or $z_2 = 1$ or $z_2 = \bar{z}_1$. The proof of the first part of Lemma 2 is now complete.

To prove the second part we note that $p'_\alpha(A\zeta + a/2)$ has the zeros

$$\zeta_1, \zeta_2 = \frac{-(a - 4\cos\alpha) \pm \left[(a - 4\cos\alpha)^2 - 36A^2\right]^{1/2}}{6A},$$

and that the quantity under the radical is nonpositive if $\alpha_1 \leqq \alpha \leqq \alpha_2$. Furthermore, it is easily verified that as α varies from α_1 to α_2 the points ζ_1, ζ_2 describe the boundary of the unit disk. Finally, if $0 \leqq \alpha < \alpha_1$ or $\alpha_2 < \alpha \leqq \pi$, then ζ_1 and ζ_2 are real and unequal and, since their product is unity, it follows that one of them lies in $|\zeta| < 1$. This implies the second part of Lemma 2 and concludes the proof.

We next establish

LEMMA 3. *Let* $p(z) = (z-a)(z-z_1)(z-z_2)$, *where* $0 \leqq a \leqq 1$, $|z_1| < 1$, *and* $|z_2| < 1$. *Then* $p'(z)$ *has at least one zero inside* $C(a)$.

PROOF. Since $|z_1| < 1$ and $|z_2| < 1$, there exists a number ρ $(0 < \rho < 1)$ such that

$$q(w) \equiv p(\rho w + (1 - \rho)a) = \rho^3(w - a)(w - w_1)(w - w_2),$$

where $|w_1| = 1$ and $|w_2| \leqq 1$. By Lemma 2, $q'(w)$ has a zero in $|w - a/2| \leqq A$ and hence $p'(z)$ has a zero $(\neq 1)$ in the disk

$$\gamma(\rho, a) \colon \left| z - a/2 - (1 - \rho)a/2 \right| \leqq \rho A.$$

Since $A \geqq a/2$ with equality only for $a=1$, it is readily verified that every point of $\gamma(\rho, a)$, except $z=1$ in the case $a=1$, lies inside $C(a)$. This proves the lemma.

Combining Lemmas 2 and 3 we clearly obtain the first part of Theorem 1 for $a>0$. For $a=0$ the required result is an immediate consequence of the fact that the modulus of the product of the zeros of $p'(z)$ is less than or equal to $1/3$.

To prove the second part of Theorem 1 for the case $a>0$ it suffices to show that for any point z_0 inside $C(a)$ there exists $p(z)\in\mathcal{P}(a, 3)$ such that $p'(z_0)=0$ and $p'(z)\neq 0$ for all z ($\neq z_0$) in $\mathfrak{D}(a, 3)$. Consider first the polynomial

$$p(z, t) = (z - a)(z - e^{i\theta})(z - \beta(t)),$$

where $\beta(t)=(1-t)a+te^{i\theta}$ $(0\leqq t\leqq 1)$. For $t=0$ we have $p(z, 0) =(z-a)^2(z-e^{i\theta})$ so that $p'(z, 0)$ has a zero at a and at $(a+2e^{i\theta})/3$. It is easy to see that the second zero lies outside $C(a)$ except for the trivial case $a=e^{i\theta}=1$. As t varies continuously from 0 to 1, each root of $p'(z, t)$ varies continuously on the line segment $[a, e^{i\theta}]$, one from a to $(2a+e^{i\theta})/3$, the other from $(a+2e^{i\theta})/3$ to $e^{i\theta}$ (compare [6, p. 24]). It follows, therefore, that no point z_0 of the closed disk $|z-2a/3| \leqq 1/3$ can be omitted from $\mathfrak{D}(a, 3)$.

Suppose next that z_0 is a point inside $C(a)$ for which $|z_0-2a/3| >1/3$, and choose z_0' $(|z_0'|>1)$ such that the derivative of $P_0(z) \equiv(z-a)(z-z_0')^2$ has a zero at z_0. We note that

$$\frac{P_0'(z_0)}{P_0(z_0)} = \frac{1}{z_0 - a} + \frac{2}{z_0 - z_0'} = 0. \tag{3}$$

Now consider the mapping $l(z)\equiv 1/(z_0-z)$. Under $w=l(z)$ the unit circumference is mapped onto a circle Γ and z_0' is mapped to a point w_0' inside Γ. Choose w_1', w_2' on Γ such that $w_0'=(w_1'+w_2')/2$, and let z_1', z_2' be the points on $|z|=1$ that satisfy $w_1'=l(z_1')$, $w_2'=l(z_2')$. It follows that

$$\frac{2}{z_0 - z_0'} = \frac{1}{z_0 - z_1'} + \frac{1}{z_0 - z_2'}, \tag{4}$$

and, combining (3) and (4), we deduce that the derivative of $p_0(z) \equiv(z-a)(z-z_1')(z-z_2')$ has a zero at $z=z_0$. Since $|z_1'|=|z_2'|=1$ implies that $|z_1'z_2'-a^2/4|\geqq 3A^2$, it is easily seen by considering the polynomial $p_0'(A\zeta+a/2)$ that the second zero of $p_0'(z)$ lies outside $C(a)$. This proves Theorem 1 for $a>0$.

For the case $a=0$ we observe that the derivative of

$$z(z - re^{i\alpha})(z - re^{i(\alpha+\pi/3)}), \qquad 0 \leqq \alpha \leqq 2\pi, \quad 0 \leqq r \leqq 1,$$

has a double zero at $z=re^{i(\alpha+\pi/6)}/\sqrt{3}$, and hence no point may be omitted from $\mathfrak{D}(0, 3)$. This completes the proof of Theorem 1.

3. **The logarithmic derivative.** By applying the methods of §2 it is possible to prove

THEOREM 2. *Let m, m_1, and m_2 be nonnegative real numbers with $n\equiv m+m_1+m_2>0$, and suppose*

$$F(z) = \frac{m}{z-a} + \frac{m_1}{z-z_1} + \frac{m_2}{z-z},$$

where $0\leqq a\leqq 1$, $|z_1|\leqq 1$, and $|z_2|\leqq 1$. Set

$$\alpha(m, n) \equiv (n-m)a/(n+m), \qquad A(m, n) \equiv (m(1-\alpha(m, n)^2)/n)^{1/2}.$$

Then $F(z)$ has at least one zero in the disk

$$|z-\alpha(m, n)| \leqq A(m, n). \tag{5}$$

We note that if $p(z)$ is a polynomial of the form

$$p(z) = (z-a)^m(z-z_1)^{m_1}(z-z_2)^{m_2},$$

where $0\leqq a\leqq 1$, $|z_1|\leqq 1$, and $|z_2|\leqq 1$, then Theorem 2 implies that $p(z)$ has at least one critical point distinct from a in the disk (5). It follows easily from this that Ilieff's conjecture is true for such $p(z)$.

4. **Conjecture.** *The equation*

$$F(z) = \frac{m}{z-a} + \sum_{k=1}^{s} \frac{m_k}{z-z_k} = 0$$

($m\geqq 0, m_k\geqq 0, n=m+m_1+\cdots+m_s>0, 0\leqq a\leqq 1, |z_k|\leqq 1$ $(1\leqq k\leqq s)$) has a root ζ satisfying

$$|\zeta-\alpha| \leqq \left\{(1-\alpha)^{s-1}\left(\frac{m}{n}+\frac{\alpha(s-1)m}{n}\right)\right\}^{1/s},$$

where $\alpha=(n-m)a/(n+(s-1)m)$.

Support for the conjecture comes from Theorem 2 (the case $s=2$). It is also easy to verify that the conjecture is true for $a=0$. Its truth for $a=1$ follows from a modification of the proof in [2].

We remark that for the case where $m=1$, $m_k=1$ $(1\leqq k\leqq s)$, the conjecture is due to J. S. Ratti and is sharper than Ilieff's.

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References

1. W. K. Hayman, *Research problems in function theory*, Athlone Press, London, 1967. MR **36** #359.

2. A. W. Goodman, Q. I. Rahman and J. S. Ratti, *On the zeros of a polynomial and its derivative*, Proc. Amer. Math. Soc. 21 (1969), 273–274. MR **39** #421.

3. G. Schmeisser, *Bermerkungen zu einer Vermutung von Ilieff*, Math. Z. **111** (1969), 121–125.

4. D. A. Brannan, *On a conjecture of Ilieff*, Proc. Cambridge Philos. Soc. **64** (1968), 83–85. MR **36** #3958.

5. A. Joyal, *On the zeros of a polynomial and its derivative*, J. Math. Anal. Appl. **26** (1969), 315–317. MR **39** #420.

6. J. L. Walsh, *The location of critical points of analytical and harmonic functions*, Amer. Math. Soc. Colloq. Publ., vol. 34, Amer. Math. Soc., Providence, R. I., 1950. MR **12**, 249.

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