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ON THE IDEAL OF UNCONDITIONALLY CONVERGENT FOURIER SERIES IN $L_p(G)$

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ABSTRACT. Let G be a compact abelian group. We consider the ideal of functions in $L_p(G)$ with unconditionally convergent Fourier series in the L_p norm. This ideal is shown to coincide with the "Derived algebra" of Helgason. A characterization of this ideal is given when p is an even integer.

Let G be a compact abelian group with dual group Γ and with normalized Haar measure (so that $f\to\hat{f}$ is an isometry of $L_2(G)$ onto $l_2(\Gamma)$). For $1\leqq p\leqq\infty$, $L_p(G)$ is a Banach algebra with convolution for multiplication. Let S_p denote the ideal of functions in L_p having unconditionally convergent Fourier series in the L_p norm.

In §2 we show that, for the Banach algebra L_p, S_p coincides with the "Derived algebra" of Helgason [7]; from this and [7] one immediately obtains that $S_p=L_2$ for $1\leqq p<2$. This is a special case of a theorem of Grothendieck [4].

In §3 we show that, for p an even integer, if $f\in L_p$ and $\hat{f}\geqq 0$ then then $f\in S_p$. From this it follows that (for p an even integer) S_p coincides with the set of $f\in L_p$ such that $|\hat{f}|$ is the transform of an L_p function. Thus for such p one can characterize the Rudin $\Lambda(p)$ sets as those subsets $E\subset\Gamma$ for which $|\hat{f}|$ is the transform of an L_p function whenever $f\in L_p$ and $\hat{f}=0$ outside of E.

1. **Preliminaries.** For facts we use about harmonic analysis the reader is referred to [3] and [9], for unconditional convergence and related notions, to [2], and for $L_p(G)$ and dual algebras, to [8].

Let $\mathfrak{F}$ denote the set of finite subsets of Γ.

THEOREM 1. *Let $1\leqq p\leqq\infty$. Then S_p is a semisimple dual Banach algebra with the norm given by*

$$\|f\|_{S_p}=\sup_{J\in\mathfrak{F}}\left\|\sum_J\hat{f}(\gamma)\gamma\right\|_p.$$

If $p=\infty$, then S_p is isomorphic and homeomorphic to $l_1(\Gamma)$, where the latter has pointwise multiplication.

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PROOF. (See [1] for details.) One verifies that S_p is a Banach algebra and that each $f \in S_p$ has an unconditionally convergent Fourier series in the S_p norm. Thus each closed ideal in S_p is the closed linear span of the characters it contains. From this it follows that S_p is a semisimple dual algebra.

The second assertion is straightforward to verify.

2. S_p and the Derived algebra.

DEFINITION (HELGASON). For $1 \leqq p < \infty$, define the Derived algebra, D_p, to be the set of $f \in L_p$ such that

$$\sup_{g \in L_p} \frac{\|f * g\|_p}{\|\hat{g}\|_\infty} \equiv \|f\|_{D_p} < \infty.$$

After the following lemma, we will show that D_p and S_p are the same Banach algebra.

LEMMA 1. *Every element of $l_\infty(\Gamma)$ is a multiplier for S_p.*

PROOF. We want to show that, given $f \in S_p$, $a \in l_\infty(\Gamma)$, there exists $g \in S_p$ such that $\hat{g} = a\hat{f}$. This follows from the fact that unconditional convergence implies bounded-multiplier convergence in a Banach space [2, p. 59], [5, p. 92].

THEOREM 2. *The Banach algebras S_p and D_p coincide, and $\|\cdot\|_{S_p} \leqq \|\cdot\|_{D_p}$.*

PROOF. By Theorem 2 of [7], D_p is the set of functions in L_p for which every element of $c_0(\Gamma)$ is a multiplier. Thus by the lemma, $S_p \subseteq D_p$.

If $f \in L_p$, $J \in \mathfrak{F}$, and $g = \sum_J \gamma$, then $\|\hat{g}\|_\infty = 1$ and $f * g = \sum_J \hat{f}(\gamma)\gamma$. Thus $D_p \subseteq S_p$[1] and $\|\cdot\|_{S_p} \leqq \|\cdot\|_{D_p}$. By the Closed Graph Theorem, the two norms are equivalent.

COROLLARY (GROTHENDIECK). *Let $1 \leqq p < 2$. If $f \in L_p$ and f has an unconditionally convergent Fourier series in the L_p norm, then $f \in L_2$.*

PROOF. This follows directly from the above theorem and [7, Theorems 5 and 7].

3. A characterization of S_p and the Rudin $\Lambda(p)$ sets when p is an even integer.

We first prove the following about functions with nonnegative transform:

THEOREM 3. *Let $p = 2k$, k an integer, and let $f \in L_p$ such that $\hat{f} \geqq 0$.*

[1] This follows from the Orlicz-Pettis Theorem [2, p. 60].

Then

(1) *f has an unconditionally convergent Fourier series in the L_p norm;*

(2) $\|f\|_p = \|f\|_{D_p} = \|f\|_{S_p}$.

PROOF. We claim that it is sufficient to show that

$$\sup\{\|f * g\|_p \mid g \text{ a trigonometric polynomial, } \|\hat{g}\|_\infty \leqq 1\} \leqq \|f\|_p.$$

Since the trigonometric polynomials are dense in L_p, we then deduce that

$$\|f * h\|_p \leqq \|f\|_p \|\hat{h}\|_\infty, \qquad h \in L_p.$$

Thus $f \in D_p = S_p$, so (1) follows. Now the above inequality gives that $\|f\|_{D_p} \leqq \|f\|_p$. Since $\|f\|_p \leqq \|f\|_{S_p} \leqq \|f\|_{D_p}$, (2) follows as well.

So, let $g = \sum_{i=1}^n a_i\gamma_i$, where $|a_i| \leqq 1$, and let

$$h = f * g = \sum_{i=1}^n \hat{f}(\gamma_i) a_i \gamma_i.$$

Letting "$[k]$" denote kth convolution power, we have that

$$\|h\|_p^p = \int |h^k|^2 = \|\widehat{h^k}\|_2^2 = \|\hat{h}^{[k]}\|_2^2 \leqq \||\hat{h}|^{[k]}\|_2^2.$$

Since $|\hat{h}| \leqq \hat{f}$, we have that

$$\||\hat{h}|^{[k]}\|_2^2 \leqq \|\hat{f}^{[k]}\|_2^2 = \|\widehat{f^k}\|_2^2 = \int |f|^{2k} = \|f\|_p^p.$$

Therefore $\|h\|_p \leqq \|f\|_p$, and the theorem is established.

As a corollary, we have the following characterization of S_p:

THEOREM 4. *Let p be an even integer, and let $f \in L_p$. Then the following statements are equivalent:*

(1) *f has an unconditionally convergent Fourier series in the L_p norm.*

(2) *$|\hat{f}|$ is the Fourier transform of an L_p function.*

PROOF. Lemma 1 gives that (1) implies (2). If (2) holds and $\hat{g} = |\hat{f}|$, then $g \in S_p$ by the above theorem; so $f \in S_p$ by Lemma 1.

REMARKS. (1) For G a compact abelian group, Bochner's theorem [9, p. 19] may be stated as follows:

If ϕ is continuous on G and $\hat{\phi} \geqq 0$, then $\phi \in S_\infty$. Thus Theorem 3 is an L_p analog of Bochner's theorem, for p an even integer.

(2) The essential ingredient in the proof of Theorem 3 is the fact that, for p an even integer, if one multiplies the coefficients of a trigonometric polynomial by complex numbers of absolute value one, the largest L_p norm is obtained when the coefficients become non-negative. This was first observed by Hardy and Littlewood [6, p.

305]. It is an open question as to what the maximum is for other values of p.

For $E \subset \Gamma$, $1 < p < \infty$, let L_p^E be the set of functions in $L_p(G)$ whose transforms vanish outside of E. The set E is said to be of type $\Lambda(p)$ if there exists $q < p$ such that $L_p^E = L_q^E$. For a discussion of equivalent notions, the reader is referred to [3].

The following lemma is known.

Lemma 2. *Let* $2 < p < \infty$. *Then* E *is of type* $\Lambda(p)$ *if and only if* $L_p^E \subset S_p$.

Proof. If E is of type $\Lambda(p)$, then $L_p^E = L_2^E$, so $L_p^E \subset S_p$.

Suppose conversely that $L_p^E \subset S_p$. Let $f \in L_2^E$. Then (cf. [3, 14.3.2]) for some $a : \Gamma \to \{-1, 1\}$, the function $a\hat{f}$ is the transform of a function in L_p^E. By Lemma 1, $a(a\hat{f}) = \hat{f}$ is also the transform of an L_p^E function. Therefore $L_p^E = L_2^E$, so E is of type $\Lambda(p)$:

As a direct consequence of the above lemma and Theorem 4, we have the following characterization of sets of type $\Lambda(p)$:

Theorem 5. *Let* $p > 2$ *be an even integer and let* $E \subset \Gamma$. *Then* E *is of ype* $\Lambda(p)$ *if and only if* $|\hat{f}|$ *is the transform of an* L_p^E *function whenever* $\in L_p^E$.

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