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ELEMENTARY PROOF OF A THEOREM OF HELSON

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ABSTRACT. An elementary proof is given of a theorem of Helson, to the effect that in a locally compact abelian group, the transform of a measure concentrated on a Helson set dces not vanish at infinity.

G is a locally compact abelian group whose dual is denoted Γ. A compact set H in G is called a Helson set if to every $F \in C(H)$ there corresponds an $f \in L^1(\Gamma)$ such that

$$F(x) = \int_\Gamma \overline{(x, \gamma)} f(\gamma) d\gamma, \qquad x \in H.$$

A simple characterization of Helson sets is the following (see e.g. [**3**, p. 115]): A compact set H is a Helson set if and only if there exists a constant B such that if $\sigma \in M(H)$, i.e. if σ is a measure concentrated on H, then

(1) $$\|\sigma\| \leqq B \|\hat{\sigma}\|_\infty.$$

Helson's theorem (see a proof in [2] for $G = R$, and a more difficult proof in [**3**, p. 119], for the general case) is the following:

Suppose H is a Helson set, $\sigma \in M(H)$ and $\sigma \neq 0$. Then $\hat{\sigma} \notin C_0(\Gamma)$, that is $\hat{\sigma}$ does vanish at infinity.

An interesting consequence of this theorem is that every Helson set has Haar measure zero.

We shall use the following well known facts:

If $\hat{\sigma} \in C_0(\Gamma)$ then σ is a continuous measure.

If $\hat{\sigma} \in C_0(\Gamma)$ and if g is any bounded Borel function, then $(g\sigma)^\wedge \in C_0(\Gamma)$. In particular if $\sigma \in M(H)$, $\hat{\sigma} \in C_0(\Gamma)$, then multiplying σ by a Borel function of modulus 1 we get a *real* measure on H whose transform still vanishes at infinity.

If σ is a real continuous measure on a set S, $\sigma(S) \neq 0$, and if N is any positive integer, then we can partition S into N disjoint sets S_i such that $\sigma(S_i) = N^{-1}\sigma(S)$ (see e.g. [**1**, p. 63, Exercise 9]).

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We shall show in an elementary way, that for a certain positive integer N, depending on the Helson set H:

$$\begin{aligned}&\text{If } \sigma \in M(H),\ \sigma \text{ real},\ \sigma \neq 0,\ \hat{\sigma} \in C_0(\Gamma), \text{ then } \exists\, \mu \in M(H):\\ &\mu \text{ real},\ \ \|\mu\| = \|\sigma\|,\ \ \hat{\mu} \in C_0(\Gamma),\ \ \|\hat{\mu}\|_\infty \leqq (1 - 3/N)\|\hat{\sigma}\|_\infty.\end{aligned} \tag{2}$$

This will contradict (1), since by repeated application it yields a sequence μ_n in $M(H)$, with $\|\mu_n\| = \|\sigma\|$, $\|\hat{\mu}\|_\infty \to 0$ and Helson's theorem will be proved.

PROOF OF (2). Choose the positive integer N such that $2N^{-1}B \leqq (1-2N^{-1})$. Let $\epsilon > 0$ be arbitrary. Let K be a compact set in Γ such that $|\hat{\sigma}(\gamma)| < \epsilon$ for $\gamma \notin K$. We can decompose H into a union of disjoint sets E_i, $i = 1, \cdots, n$, such that, for each i, (x, γ) varies very little for $x \in E_i$, $\gamma \in K$. In particular, for $x_i \in E_i$,

$$\Big| \sum_i \sigma(E_i)\overline{(x_i, \gamma)} - \hat{\sigma}(\gamma) \Big| < \epsilon \quad \text{for } \gamma \in K. \tag{3}$$

Each E_i contains a set E_i' such that $\sigma(E_i') = N^{-1}\sigma(E_i)$ and we can choose E_i' such that

$$|\sigma|(E_i') \leqq N^{-1}|\sigma|(E_i). \tag{4}$$

Let χ' be the characteristic function of $H' = \bigcup E_i'$ and put $\sigma' = \chi'\sigma$. By our choice of the E_i we may suppose that, for $x_i \in E_i$,

$$\Big| \sum_i \sigma'(E_i')\overline{(x_i, \gamma)} - \hat{\sigma}'(\gamma) \Big| < \epsilon \quad \text{for } \gamma \in K. \tag{3'}$$

Put $\mu = (1 - 2\chi')\sigma$. Since $|1 - 2\chi'| = 1$ we have $\|\mu\| = \|\sigma\|$. Also $\hat{\mu} \in C_0(\Gamma)$.

Now for $\gamma \in K$, we have, by (3) and (3′)

$$\begin{aligned}|\hat{\mu}(\gamma)| &= |\hat{\sigma}(\gamma) - 2\hat{\sigma}'(\gamma)| \leqq 3\epsilon + \left(1 - \frac{2}{N}\right)\Big|\sum_i \sigma(E_i)\overline{(x_i, \gamma)}\Big| \\ &\leqq 3\epsilon + \left(1 - \frac{2}{N}\right)(\|\hat{\sigma}\|_\infty + \epsilon) \leqq \left(1 - \frac{3}{N}\right)\|\hat{\sigma}\|_\infty \qquad (\epsilon \text{ small})\end{aligned}$$

while for $\gamma \notin K$, by (4),

$$\begin{aligned}|\hat{\mu}(\gamma)| &< \epsilon + 2\|\sigma'\| \leqq \epsilon + \frac{2}{N}\|\sigma\| \leqq \epsilon + \frac{2}{N}B\|\hat{\sigma}\|_\infty \\ &\leqq \epsilon + \left(1 - \frac{2}{N}\right)\|\hat{\sigma}\|_\infty \leqq \left(1 - \frac{3}{N}\right)\|\hat{\sigma}\|_\infty.\end{aligned}$$

The proof of (2) is now complete.

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