PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, No. 2, February 1971

HIGHER ORDER HOMOLOGY OPERATIONS AND THE ADAMS SPECTRAL SEQUENCE

WARREN M. KRUEGER

ABSTRACT. One can filter the stable homotopy groups of spheres in such a way as to obtain the (semisimplicial) mod-p Adams spectral sequence and also by means of certain higher order homology operations. In this note we show that these filtrations coincide.

1. Introduction. To each prime p and each (semisimplicial) spectrum X of finite type, Bousfield et al. [2] associate a spectral sequence $E(X, p)=\{E^iX, d^iX\}$ which converges to the p-primary component of π_*X. The filtration giving the convergence is that induced by a subsequence of the mod-p lower central series of FX the free group spectrum on X, namely

$$F_s\pi_mX = \operatorname{im}\{\pi_m\Gamma_{p^s}FX \to \pi_mFX\}.$$

In the event that $X=S$, the spectrum of spheres, π_*S admits another filtration which can be described by means of higher order homology operations. The only operations considered here shall be those that arise from P-systems for which $A_0=KZ_p$ and each A_i is a Z_p-module spectrum [3, §II.5]. For each $\alpha\in\pi_mS$, let $X(\alpha)$ be the spectrum for which $(X(\alpha))_n=S^n U_\alpha\Delta^{m+n+1}$. Then $H_0(X(\alpha);\ Z_p)=H_{m+1}(X(\alpha);\ Z_p)=Z_p$ as long as $h\alpha=0$ where $h:\pi_*S\to H_*(S;\ Z_p)$ is the Hurewicz map. Define $F_s'\pi_mS$ to be the subgroup of π_mS consisting of those elements α meeting the following condition: each homology operation Φ of order $r<s$ is defined and is zero on $H_*(X(\alpha);\ Z_p)$. The definition is completed by setting $F_1'\pi_mS=\{\alpha\in\pi_mS \mid h\alpha=0\}$.

The purpose of this note is to show that the two filtrations described above coincide for $X=S$. In [1], Adams conjectured that this was the case for the topological version of this spectral sequence.

The author wishes to thank the referee for suggesting to him remark (c) of §3.

2. Proof that $F_s\pi_mS=F_s'\pi_mS$. The proof is an immediate consequence of the following two lemmas.

Received by the editors November 19, 1969 and, in revised form, April 4, 1970.

AMS 1969 *subject classifications*. Primary 5534, 5552.

Key words and phrases. Semisimplicial spectrum, Adams spectral sequence, P-system, homology operation, universal homology operation, n-step nilpotent spectrum, n-step metabelian spectrum.

LEMMA 1. *For each r, let Φ_r be the homology operation associated to the P-system*

$$\begin{tikzcd}
(\Gamma_1/\Gamma_p)FS \arrow[d, equal] & & (\Gamma_{p^r}/\Gamma_{p^{r+1}})FS \arrow[d] \\
(\Gamma_1/\Gamma_p)FS & \cdots \arrow[l] & (\Gamma_1/\Gamma_{p^{r+1}})FS \arrow[l]
\end{tikzcd}$$

Then $\alpha \in F_{s+1}\pi_m S$ if and only if Φ_r is defined and zero on $H_*(X(\alpha); Z_p)$ for all $r < s+1$.

LEMMA 2. *The operations Φ_r are universal in the sense of Kan-Whitehead* [**3**, §II.6] *for all r.*

PROOF OF LEMMA 1. As $X(\alpha)$ has nontrivial homology only in dimensions 0 and $m+1$, only the component of Φ_r of degree $m+1$ acting on $H_{m+1}(X(\alpha); Z_p)$ need be considered. The argument is given in the case that $r=s$. One first observes that $H_*(X; (\Gamma_1/\Gamma_{p^{s+1}})FS) \approx \pi_*\Gamma_1/\Gamma_{p^{s+1}}FX$ for all spectra X. Then segments of the exact homotopy sequences associated to the fiber maps j, j', k, k', q and q' can be amalgamated into the following commutative diagram

$$\begin{tikzcd}
 & & \pi_m\Gamma_{p^{s+1}}FS \arrow[d, "i_*"] \\
 & Z \approx \pi_{m+1}F(X(\alpha)\backslash S) \arrow[r, "\partial"] \arrow[d, two heads, "q_*"] & \pi_m FS \arrow[d, "q'_*"] \\
\pi_{m+1}(\Gamma_1/\Gamma_{p^{s+1}})FX(\alpha) \arrow[r, "j_*"] \arrow[d, "k_*"'] & \pi_{m+1}(\Gamma_1/\Gamma_{p^{s+1}})F(X(\alpha)\backslash S) \arrow[r, "\partial"] \arrow[d, two heads, "k'_*"] & \pi_m(\Gamma_1/\Gamma_{p^{s+1}})FS \\
Z_p \approx \pi_{m+1}(\Gamma_1/\Gamma_p)FX(\alpha) \arrow[r, "j'_*", "\approx"'] & \pi_{m+1}(\Gamma_1/\Gamma_p)F(X(\alpha)\backslash S) \approx Z_p &
\end{tikzcd}$$

in which the middle row and far right column are exact. As

$$\pi_m\Gamma_{p^{s+1}}F(X(\alpha)\backslash S) = 0 \quad \text{and} \quad \pi_{n+1}(\Gamma_1/\Gamma_{p^{s+1}})F(X(\alpha)\backslash S) = 0,$$

the maps q_* and k'_* indicated as epimorphisms are such. Moreover, the groups $\pi_m(\Gamma_1/\Gamma_{p^{s+1}})FS$ and $\pi_{m+1}(\Gamma_1/\Gamma_{p^{s+1}})F(X(\alpha)\backslash S)$ are finite and p-primary and in fact the second group is cyclic of order p^e, $e \geqq 1$.

Let β be the generator of $\pi_{m+1}F(X(\alpha)\backslash S)$ for which $\partial\beta = \alpha$, ν be the generator $q_*\beta$ of $\pi_{m+1}(\Gamma_1/\Gamma_{p^{s+1}})F(X(\alpha)\backslash S)$, ξ' be the generator $k'_*q'\beta$ of $\pi_{m+1}(\Gamma_1/\Gamma_p)F(X(\alpha)\backslash S)$ and ξ be the generator $(j')^{-1}(\xi')$ of $\pi_{m+1}(\Gamma_1/\Gamma_p)FX(\alpha)$. By the remark on p. 246 of [**3**], Φ_s vanishes on $H_*(X(\alpha); Z_p)$ if and only if there is $\zeta \in \pi_{m+1}(\Gamma_1/\Gamma_{p^{s+1}})FX(\alpha)$ for which $k_*\zeta = \xi$. On the other hand, $\alpha \in F_s\pi_m S$ if and only if $\partial\nu = 0$. Thus to prove the lemma, it suffices to show that there is a class ζ such that $k_*\zeta=\xi$ if and only if $\partial\nu=0$.

Suppose that $\partial\nu=0$; by exactness there is $\zeta\in\pi_{m+1}(\Gamma_1/\Gamma_{p^{s+1}})FX(\alpha)$ so that $j_*\zeta=\nu$. But commutativity of the lower left-hand square of the diagram and the fact that j'_* is an isomorphism imply that $k_*\zeta=\xi$. Conversely suppose that there is a class ζ with $k_*\zeta=\xi$ but that $\partial\nu\neq 0$. Now $j_*(\zeta)=t\nu$ for some integer t such that $1<t<p^e$. As $0=\partial(t\nu)=t\partial\nu$, t must equal p^f for some f with $1\leqq f<e$. Hence $j'_*\xi=k'_*(p^f\nu)=p^f k'_*\nu=0$, a fact which contradicts the isomorphy of j'_*.

Proof of Lemma 2. The proof here proceeds exactly as that of Theorem 6.5 of [3] with one modification. As the operations of (6.5) arise from the derived series of FS rather than the p-lower central series, analogues of Definition 3.1 and Lemma 3.4 of [3] are needed in the present setting.

Definition. A spectrum X is said to be n-step nilpotent (modulo p) if the inclusion $\Gamma_p^n FX\to FX$ is null-homotopic.

Proposition. *If $X\xrightarrow{i}Y\xrightarrow{q}Z$ is a fibration with X m-step nilpotent and Z n-step nilpotent, then X is $(m+n)$-step nilpotent.*

Proof. As in (3.4) of [3], one has a diagram

$$\begin{tikzcd}
 & \Gamma_{p^{m+n}}FY \arrow[dl, dashed, "g_m"'] \arrow[d, hook, "i_4"] & \\
\Gamma_{p^m}FX \arrow[d, hook, "i_1"] & \Gamma_{p^n}FY \arrow[r, "\Gamma_{p^n}Fq"] \arrow[d, hook] \arrow[dl, dashed, "g"'] & \Gamma_{p^n}FZ \arrow[d, hook, "i_3"] \\
FX \arrow[r, "Fi"] & FY \arrow[r, "Fq"] & FZ
\end{tikzcd}$$

in which i_1 and i_3 are null-homotopic and the rectangle is commutative. The fact that $i_3\simeq *$ implies that there is a map $g:\Gamma_{p^n}FY\to FX$ so that $(Fi)g\simeq i_2$. If there is a map $g_m:\Gamma_{p^{m+n}}FY\to\Gamma_{p^m}FX$ for which $i_1g_m\simeq gi_4$, then $i_2i_4\simeq *$, since it factors through i_1 up to homotopy. Unlike (3.4), g_m cannot simply be taken to be $\Gamma_{p^m}g$, since $\Gamma_{p^m}(\Gamma_{p^n}FY)\subsetneqq\Gamma_{p^{m+n}}FY$ in general. However, it is not hard to show that g_m can be obtained from $g|\Gamma_{p^{m+n}}FY$ by deformation. To see that g_m exists, one proceeds inductively; assume the existence of g_s and set $g'_{s+1}=g_s|\Gamma_{p^{n+s+1}}FY\to\Gamma_{p^s}FX$. Now g'_{s+1} can be deformed into $\Gamma_{p^{s+1}}FX$ if and only if its associated map $\Gamma_{p^{n+s+1}}FY\to(\Gamma_{p^s}/\Gamma_{p^{s+1}})FX$ is null-homotopic and this obtains if and only if

$$H^*((\Gamma_{p^s}/\Gamma_{p^{s+1}})FX;\ Z_p) \to H^*(\Gamma_{p^{n+s+1}}FY:Z_p)$$

is zero. But this homomorphism factors through the inclusion induced homomorphism $H^*(\Gamma_{p^{n+s}}FY;\ Z_p) \to H^*(\Gamma_{p^{n+s+1}}FY;\ Z_p)$ which is zero by §7 of [**2**]. Thus g'_{s+1} is homotopic to $g_{s+1}:\Gamma_{p^{n+s+1}}FY \to \Gamma_{p^{s+1}}FX$.

3. **Concluding remarks.** (a) If $p=2$ then $\Phi_1=\sum Sq^i$ the big Steenrod square; this follows easily from the structure of the differential d^1 on E^1S [**2**].

(b) The equality of these two filtrations identifies the classical technique of locating nontrivial elements in the stems by means of detection with cohomology operations as an aspect of computing with the Adams spectral sequence.

(c) It is not hard to show that a spectrum X is n-step nilpotent (mod-p) if and only if X is *n-step metabelian* (mod-p) where this latter notion is defined exactly as in Kan-Whitehead [**3**, p. 241], but using the mod-p derived series.

References

1. J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180–204. MR **20** #2711.

2. A. K. Bousfield, et al, *The* mod-p *lower central series and the Adams spectral sequence*, Topology **5** (1966), 331–342. MR **33** #8002.

3. D. M. Kan and G. W. Whitehead, *Orientability and Poincaré duality in general homology theories*, Topology **3** (1965), 231–270. MR **32** #8335.

New Mexico State University, Las Cruces, New Mexico 88001