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ONE-SIDED BOUNDARY BEHAVIOR FOR CERTAIN HARMONIC FUNCTIONS

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ABSTRACT. Some results concerning the maximal ideal space of H^∞ of the disk are applied to harmonic functions. The methods yield a Lindelöf type theorem for harmonic functions and extend to bounded harmonic functions a criterion of Tanaka which is necessary and sufficient in order that the boundary value function be one-sided approximately continuous.

1. **Introduction.** We are concerned in this paper with connections between the one-sided behavior of an L^∞ function at a point of the unit circle, $C=\{z:|z|=1\}$, and the boundary behavior of the harmonic extension of the function into the unit disc, $D=\{z:|z|<1\}$. Our techniques consist mainly of combining certain concrete estimates for harmonic measures with some facts about the Banach algebra, H^∞, of bounded analytic functions on D. We assume for this latter area that the reader is familiar with the contents of Chapter 10 of Hoffman's book, [**3**], and with [**4**] and [**5**].

The main focal point is the introduction of a class of homomorphisms in the maximal ideal space of H^∞ which we call the "barely tangential homomorphisms." These homomorphisms play a role for the one-sided boundary behavior of L^∞ functions similar to that played by the radial homomorphisms for two-sided behavior in [**1**].

§2 is mainly devoted to an intrinsic study of the barely tangential homomorphisms. In §3 we obtain a theorem (Theorem 3.1) characterizing one-sided approximate continuity from above of an L^∞ function in terms of the behavior of the function on the supports of the representing measures of upper barely tangential homomorphisms. Subsequently, we easily obtain a result of Tanaka [**6**, Theorem 5] characterizing one-sided approximate continuity. As a final result we prove a "Lindelöf-type theorem" for L^∞ functions.

2. **Barely tangential homomorphisms.** We first recall that the collection, H^∞, of bounded analytic functions on D forms a function algebra with pointwise operations and the supremum norm. Its maximal ideal space, $\mathfrak{D}$, is a compactification of D, [**3**]. Every homomorphism in $\mathfrak{D}$ can be approached by a universal net in D or by one

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in any dense subset of D. One can also represent H^∞ as a subalgebra of the Banach algebra, L^∞, of all bounded measurable functions on C. We rely on the references for a complete description of these connections. We remark here that for any L^∞ function f on C we will continue to denote as "f" any of the standard representations of f on D, $\mathfrak{D}$, or C.

For simplicity we restrict our attention to the fiber, $\mathfrak{D}_1$, above 1, i.e., those homomorphisms which are approached by nets tending to 1. From here on out we will simply assume phrases such as, "at 1." The collection, $\mathcal{S}$, of such homomorphisms which are approached within a Stolz angle are called the *Stolz homomorphisms*. In [**5**] the w^*-closure, $\mathcal{S}^- = \mathcal{L}$, of the Stolz homomorphisms is called the *Lindelöf homomorphisms.*

Definition 2.1. Let $\mathcal{S}$ and $\mathcal{L}$ be the Stolz and Lindelöf homomorphisms, respectively. Then, the collection, $\mathcal{B} = \mathcal{L} - \mathcal{S}$, is called the *barely tangential homomorphisms.* Those points, $\mathcal{B}^+$ [$\mathcal{B}^-$], of $\mathcal{B}$ which are approached by nets tending to 1 tangentially from above [below] are termed the *upper* [lower] *barely tangential homomorphisms.*

Our first result shows a relationship between $\mathcal{B}$ and $\mathcal{L}$ which we shall use once in §3. We recall that for any function algebra, A, on a compact Hausdorff space X, each point $h \in X$ has a representing measure, μ_h, spread on the Šilov boundary of A.

Lemma 2.2. *Let A denote the restriction algebra of H^∞ to the Lindelöf homomorphisms, $\mathcal{L}$. Then, the Šilov boundary of A is contained in the barely tangential homomorphisms, $\mathcal{B}$. If h is a Stolz homomorphism, then for any representing measure μ_h we have $\mu_h(\mathcal{B}^+) > 0$ and $\mu_h(\mathcal{B}^-) > 0$. In particular, if $f \in H^\infty$, then f has radial limit α if and only if f is constantly α on $\mathcal{B}^+$ (or on $\mathcal{B}^-$).*

Proof. We show that no Stolz homomorphism is contained in the Šilov boundary. Let $h \in \mathcal{S}$. Since by [**5**, §3] $\mathcal{S}$ is open in $\mathfrak{D}_1$ and since $\mathcal{B}$ is compact there is a neighborhood N of h such that $N \subset \mathcal{S}$ and $N \cap \mathcal{B} = \emptyset$. Now, suppose $f \in H^\infty$ and f peaks on N, i.e., $|f(h_1)| = \|f\|_{\mathcal{L}}$ for some $h_1 \in N$. Choose some Gleason part, P, of $\mathfrak{D}$ which contains such a point in N. Using the results of [**5**, (especially §6)] we see that $P \subset \mathcal{S}$ and $P^- \cap \mathcal{B} \neq \emptyset$. Since the restriction of f to P is analytic and achieves its maximum modulus on P it is constant. Thus f peaks at some point of $P^- \cap \mathcal{B}$. Consequently, we have shown that f peaks outside N. Therefore, h cannot be in the Šilov boundary and the latter is contained in $\mathcal{B}$.

Next, let $h \in \mathcal{S}$ and let μ_h be a representing measure. Let u be the harmonic measure of $C^+ = \{e^{i\theta} : 0 \leqq \theta \leqq \pi\}$. Let v be a harmonic conjugate and let $f=\exp(u+iv)$. Then $|f|=e$ on $\mathcal{B}^+$, $|f|=1$ on $\mathcal{B}^-$ and $1<|f|<e$ on $\mathcal{L}$. If $\mu_h(\mathcal{B}^+)=0$, then we would have $|f(h)|\leqq 1$ which is impossible. Thus $\mu_h(\mathcal{B}^+)>0$, and similarly $\mu_h(\mathcal{B}^-)>0$.

Finally, suppose $f\in H^\infty$. If f has radial limit α, then f has Stolz limit α and thus $f=\alpha$ on $\mathcal{S}$. Consequently, by continuity $f=\alpha$ on $\mathcal{B}^+$ (or $\mathcal{B}^-$). Conversely, suppose $f=\alpha$ on $\mathcal{B}^+$ (or $\mathcal{B}^-$). Let μ_h be a Jensen measure for $h\in\mathcal{S}$. Then,

$$\log|f(h)-\alpha| \leqq \int_{\mathcal{B}^+\cup\mathcal{B}^-}\log|f-\alpha|\,d\mu_h.$$

Since $\mu_h(\mathcal{B}^+)>0$ and $f-\alpha=0$ on $\mathcal{B}^+$, the integral above equals $-\infty$ and so $f(h)=\alpha$. Thus, $f=\alpha$ on $\mathcal{S}$ and *a fortiori* f tends to α radially.

In order to avoid exactly similar cases we now concentrate entirely on the upper barely tangential homomorphisms.

We next define a cluster set which will allow us to state more concretely Theorem 3.3 in §3. Each Stolz angle approach can be written as $\theta=c(1-r)$ and as c increases the approach tends more and more toward an upper tangential approach. Given $d\leqq c$ we denote by $\mathcal{S}_{d,c}$ the collection of all homomorphisms in $\mathfrak{D}_1$ which are approached by nets corresponding to all Stolz angle approaches, a, such that $d\leqq a\leqq c$. Given $f\in L^\infty$ define $S_{d,c}(f,1)$ as the collection of all cluster values of $f(re^{i\theta})$ as $re^{i\theta}\to 1$ and $d\leqq\liminf\theta/(1-r)\leqq\limsup\theta/(1-r)\leqq c$. It is not hard to see that we then have $S_{d,c}(f,1)=f[\mathcal{S}_{d,c}]$. Now, define $B^+(f,1)$, the *upper barely tangential cluster set* of f at 1 by

$$B^+(f,1)=\bigcap_{d\geqq 0}\left[\bigcup_{c\geqq 0}S_{d,c}(f,1)\right]^-.$$

LEMMA 2.3. *Let $f\in L^\infty$. Then, $B^+(f,1)=f[\mathcal{B}^+]$.*

PROOF. Suppose $h_0\in\mathcal{B}^+$. Then, there is a net $h_\alpha\in\mathcal{S}$ such that $h_\alpha\to h_0$. Clearly, for each $d\geqq 0$, h_α is eventually in $\bigcup_{c\geqq 0}\mathcal{S}_{d,c}$. Thus, $h_0\in\bigcap_{d\geqq 0}[\bigcup_{c\geqq 0}\mathcal{S}_{d,c}]^-$. On the other hand suppose $h_0\in\bigcap_{d\geqq 0}[\bigcup_{c\geqq 0}\mathcal{S}_{d,c}]^-$. Then, for each $d\geqq 0$, $h_0\in[\bigcup_{c\geqq 0}\mathcal{S}_{d,c}]^-$. Since $[\bigcup_{c\geqq 0}\mathcal{S}_{d,c}]^- -\mathcal{S}=\mathcal{B}^+$ for each d, $h_0\in\mathcal{B}^+$. Therefore,

$$\mathcal{B}^+=\bigcap_{d\geqq 0}\left[\bigcup_{c\geqq 0}\mathcal{S}_{d,c}\right]^-.$$

Since the intersection is over a nested system of compact sets and since f is continuous on $\mathfrak{D}$ we have

$$f[\mathcal{B}^+]=\bigcap_{d\geqq 0}\left[\bigcup_{c\geqq 0}f[\mathcal{S}_{d,c}]\right]^- = B^+(f,1).$$

Given a measurable subset, M, of the circle, C, we let u_M denote the harmonic measure of C,

$$u_M(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_M(e^{it}) \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\,dt$$

where χ_M is the characteristic function of M. If M is a subset of the upper portion of the unit circle, $C^+ = \{e^{i\theta}: 0 \leqq \theta \leqq \pi\}$, we let

$$d(M) = \liminf_{\theta\to 0^+} \frac{1}{\theta}\int_0^{\theta} \chi_M(e^{it})dt, \qquad D(M) = \limsup_{\theta\to 0^+} \frac{1}{\theta}\int_0^{\theta} \chi_M(e^{it})dt$$

denote the lower and upper densities of M at 1 from above.

Our basic results rely heavily on the following estimates for harmonic measures.

LEMMA 2.4. *Let M be a measurable subset of C^+. If $d(M) = D(M)$, then for every $h \in \mathcal{B}^+$, $u_M(h) = d(M)$.*

LEMMA 2.5. *Let M be a measurable subset of C^+. Then, there exists an $h \in \mathcal{B}^+$ with*

$$u_M(h) \geqq \frac{2}{\pi}\tan^{-1}\frac{D(M)}{2\sqrt{(1-D(M))}}\,.$$

PROOF OF 2.4. Let $\epsilon > 0$ and choose $0 < \theta_0 < \pi/2$ so that, for $0 \leqq \theta \leqq \theta_0$,

$$d(M) - \epsilon < \frac{1}{\theta}\int_0^{\theta} \chi_M(e^{it})dt \leqq d(M) + \epsilon.$$

Since the harmonic measure of $M \cap [\theta_0, \pi]$ tends to zero as $z \to 1$ we have, as $z \to 1$,

$$u_M(re^{i\theta}) = o(1) + \frac{1}{2\pi}\int_0^{\theta_0} P_r(\theta - t)\chi_M(e^{it})dt$$

where $P_r(t)$ is the Poisson kernel. Integrating by parts we have

$$\frac{1}{2\pi}\int_0^{\theta_0} P_r(\theta - t)\chi_M(e^{it})dt = \frac{P_r(\theta-\theta_0)}{2\pi}\int_0^{\theta_0} \chi_M(e^{is})ds + \frac{1}{2\pi}\int_0^{\theta_0} P_r'(\theta - t)\int_0^{t} \chi_M(e^{is})ds\,dt.$$

Thus, as $z \to 1$,

$$u_M(re^{i\theta}) = o(1) + \frac{1}{2\pi}\int_0^{\theta_0} P_r'(\theta - t)\int_0^{t} \chi_M(e^{is})ds\,dt.$$

For $0 \leqq t \leqq \theta < \pi/2$, $P_r'(\theta-t) \leqq 0$ so

$$(d(M)+\epsilon)tP_r'(\theta-t) \leqq P_r'(\theta-t)\int_0^t \chi_M(e^{is})ds \leqq (d(M)-\epsilon)tP_r'(\theta-t);$$

while, for $\theta \leqq t \leqq \theta_0$, $P_r'(\theta-t) \geqq 0$ so

$$(d(M)-\epsilon)tP_r'(\theta-t) \leqq P_r'(\theta-t)\int_0^t \chi_M(e^{is})ds \leqq (d(M)+\epsilon)tP_r'(\theta-t).$$

Therefore,

$$\begin{aligned} o(1) &+ \frac{d(M)+\epsilon}{2\pi}\int_0^\theta tP_r'(\theta-t)dt + \frac{d(M)-\epsilon}{2\pi}\int_\theta^{\theta_0} tP_r'(\theta-t)dt \\ &\leqq u_M(re^{i\theta}) \\ &\leqq o(1) + \frac{d(M)-\epsilon}{2\pi}\int_0^\theta tP_r'(\theta-t)dt + \frac{d(M)+\epsilon}{2\pi}\int_\theta^{\theta_0} tP_r'(\theta-t)dt. \end{aligned}$$

Now,

$$\int_0^\theta tP_r'(\theta-t)dt = \frac{-(1+r)\theta}{1-r} + 2\tan^{-1}\left\{\frac{1+r}{1-r}\tan\frac{\theta}{2}\right\},$$

$$\int_\theta^{\theta_0} tP_r'(\theta-t)dt = o(1) + \frac{(1+r)\theta}{1-r} + 2\tan^{-1}\left\{\frac{1+r}{1-r}\tan\frac{\theta_0-\theta}{2}\right\},$$

so that if $re^{i\theta}\to 1$ in such a way that $\theta/(1-r)\to c \geqq 0$, we have

$$\begin{aligned} (d(M)+\epsilon)&\left[\frac{-c}{\pi}+\frac{\tan^{-1}c}{\pi}\right] + (d(M)-\epsilon)\left[\frac{c}{\pi}+\frac{1}{2}\right] \\ &\leqq \liminf u_M(re^{i\theta}) \leqq \limsup u_M(re^{i\theta}) \\ &\leqq (d(M)-\epsilon)\left[\frac{-c}{\pi}+\frac{\tan^{-1}c}{\pi}\right] + (d(M)+\epsilon)\left[\frac{c}{\pi}+\frac{1}{2}\right]. \end{aligned}$$

Since this is true for every $\epsilon > 0$, we have as $re^{i\theta}\to 1$, $\theta/(1-r)\to c \geqq 0$,

$$\begin{aligned} d(M)&\left[\frac{1}{2}+\frac{\tan^{-1}c}{\pi}\right] \\ &\leqq \liminf u_M(re^{i\theta}) \leqq \limsup u_M(re^{i\theta}) \leqq d(M)\left[\frac{1}{2}+\frac{\tan^{-1}c}{\pi}\right]. \end{aligned}$$

Thus, as $re^{i\theta}\to 1$, $\theta/(1-r)\to c \geqq 0$,

$$\lim u_M(re^{i\theta}) = d(M)\left[\frac{1}{2}+\frac{\tan^{-1} c}{\pi}\right].$$

Thus, for any homomorphism, h_c, in $\mathcal{S}_{c,c}$ we have

$$u_M(h_c) = d(M)\left[\frac{1}{2}+\frac{\tan^{-1} c}{\pi}\right].$$

If $h\in\mathcal{B}^+$, it is the limit of homomorphisms h_c for which $c\to\infty$. Thus, because u_M is continuous we have, letting $c\to\infty$, $u_M(h)=d(M)$, as claimed.

PROOF OF 2.5. Let $D=D(M)$, let $\epsilon>0$, and pick $\theta_n\to 0$ such that

$$\frac{1}{2\theta_n}\int_0^{2\theta_n}\chi_M(e^{it})dt \geqq D-\epsilon.$$

Then,

$$\begin{aligned}
u_M(re^{i\theta_n}) &\geqq \int_0^{2\theta_n} P_r(\theta_n - t)\chi_M(e^{it})dt \\
&\geqq 2\int_0^{(D-\epsilon)\theta_n} P_r(\theta_n - t)dt = 2\int_{(1-D+\epsilon)\theta_n}^{\theta_n} P_r(t)dt \\
&= \frac{2}{\pi}\tan^{-1}\left\{\frac{\dfrac{1+r}{1-r}\left(\tan\dfrac{\theta_n}{2}-\tan\dfrac{(1-D+\epsilon)\theta_n}{2}\right)}{1+\left(\dfrac{1+r}{1-r}\right)^2\tan\dfrac{\theta_n}{2}\tan\dfrac{(1-D+\epsilon)\theta_n}{2}}\right\}.
\end{aligned}$$

Let r_n be determined by $\theta_n=c(1-r_n)$. Then, the limit as $\theta_n\to 0$ of the numerator of the argument of $\tan^{-1}$ in the last expression is $c(D-\epsilon)$, while that of the denominator is $1+c^2(1-D+\epsilon)$. Hence,

$$\limsup_{\theta_n\to 0} u_M(r_n e^{i\theta_n}) \geqq \frac{2}{\pi}\tan^{-1}\left\{\frac{c(D-\epsilon)}{1+c^2(1-D+\epsilon)}\right\}.$$

This being true for every $\epsilon>0$, we have

$$\limsup_{\theta_n\to 0} u_M(r_n e^{i\theta_n}) \geqq \frac{2}{\pi}\tan^{-1}\left\{\frac{cD}{1+c^2(1-D)}\right\}.$$

In particular, if we choose $c=(1-D)^{-1/2}$,

$$\limsup u_M(re^{i\theta}) \geqq \frac{2}{\pi}\tan^{-1}\left\{\frac{D}{2\sqrt{(1-D)}}\right\}, \qquad re^{i\theta}\to 1, \quad \theta = c(1-r).$$

This quantity is therefore a lower bound for $u_M(h)$ for some $h\in\mathcal{S}_{c,c}$. Therefore, by Lemma 2.2 it is also a lower bound for $u_M(h)$ for some $h\in\mathcal{B}$. Since $u_M=0$ on $\mathcal{B}^-$, it follows that for some homomorphism h in $\mathcal{B}^+$,

$$u_M(h) \geqq \frac{2}{\pi}\tan^{-1}\left\{\frac{D}{2\sqrt{(1-D)}}\right\}$$

as was to be proved.

If M is a measurable subset of C, we let $\tilde{M}=\{h\in\mathfrak{D}_1:\chi_M(h)=1\}$. We recall that the range of $f\in L^\infty$ on $\tilde{M}$ is precisely the collection of essential cluster values of $f(e^{i\theta})$ as $e^{i\theta}\to 1$ through M. We also recall the result from [1] that for each $h\in\mathfrak{D}$, $\mu_h(\tilde{M})=u_M(h)$. With these preliminaries we may now prove

COROLLARY 2.6. *Let M be a measurable subset of C^+. Then, $d(M)=1$ if and only if $\mu_h(\tilde{M})=1$ for every $h\in\mathcal{B}^+$.*

PROOF. If $d(M)=1$, we have from Lemma 2.4 that $u_M(h)=1$. From the above remark it is immediate that $\mu_h(\tilde{M})=1$. On the other hand suppose $d(M)<1$. Then, $D(\sim M)>0$ and so by Lemma 2.5 there is an $h\in\mathcal{B}^+$ with

$$u_{\sim M}(h) \geqq \frac{2}{\pi}\tan^{-1}\frac{D(\sim M)}{2\sqrt{(1-D(\sim M))}} > 0.$$

Since $u_{\sim M}+u_M=1$, $u_M(h)<1$ so $\mu_h(\tilde{M})<1$ for some $h\in\mathcal{B}^+$.

3. **Applications.** It is now an easy matter to obtain several results connecting the one-sided behavior of an L^∞ function on C at 1 with its boundary behavior at 1 from inside D.

A function f on C is ***approximately continuous from above at* 1 *with value* α** if for every $\epsilon>0$, the density $d(\{e^{i\theta}:|f(e^{i\theta})-\alpha|\leqq\epsilon,\ 0\leqq\theta\leqq\pi\})$ equals one. The main theorem upon which the applications are based is

THEOREM 3.1. *Let $f\in L^\infty$. Then f is approximately continuous from above at* 1 *with value α if and only if f is identically α on the support of the representing measure of each upper barely tangential homomorphism.*

PROOF. For each $\epsilon>0$ let $M_\epsilon=\{e^{i\theta}:|f(e^{i\theta})-\alpha|\leqq\epsilon,\ 0\leqq\theta\leqq\pi\}$. Then, f is approximately continuous from above at 1 with value α if and only if $d(M_\epsilon)=1$ for every $\epsilon>0$ if and only if, by Corollary 2.6, $\mu_h(\tilde{M}_\epsilon)=1$ for every $\epsilon>0$ and every $h\in\mathcal{B}^+$. This latter statement implies that for each $h\in\mathcal{B}^+$, the support of μ_h is contained in $\tilde{M}_\epsilon$ for every ϵ. But on $\tilde{M}_\epsilon$, $|f(h)-\alpha|\leqq\epsilon$. Thus for each $h\in\mathcal{B}^+$, f is identically α on the support of μ_h. On the other hand suppose for each $h\in\mathcal{B}^+$ that $f=\alpha$ on the support of μ_h. Then, since $|f-\alpha|\geqq\epsilon$ on $(C-M_\epsilon)^\sim$ it must be that the support of μ_h is entirely contained in $\tilde{M}_\epsilon$, i.e., $\mu_h(\tilde{M}_\epsilon)=1$. This completes the chain of implications and the theorem follows.

The next theorem is a generalization to L^∞ functions of a result of Tanaka [6, Theorem 5], for H^∞. By this time our proof is a considerable simplification of that given by Tanaka.

THEOREM 3.2. *Let $f\in L^\infty$. Then, necessary and sufficient conditions for f to be approximately continuous from above at* 1 *with value α are*

(i) *$f(h)=\alpha$ for all $h\in\mathcal{B}^+$.*

(ii) *The set, $\{e^{i\theta}:|f(e^{i\theta})|\leqq|\alpha|+\epsilon\}$, has density* 1 *at* 1 *from above for every $\epsilon>0$.*

NOTE. In Tanaka's theorem condition (i) was the statement that f tends to α radially. From Lemma 2.2 we see that this is equivalent to our (i) for H^∞ functions.

PROOF. First suppose f is approximately continuous from above at 1 with value α. By Theorem 3.1, $f=\alpha$ on the support of the representing measure for each $h\in\mathcal{B}^+$. Immediately, $f=\alpha$ on $\mathcal{B}^+$. Condition (ii) is necessary because

$$\{e^{i\theta}:|f(e^{i\theta})-\alpha|\leqq\epsilon\}\subset\{e^{i\theta}:|f(e^{i\theta})|\leqq|\alpha|+\epsilon\}.$$

Next, suppose the conditions (i) and (ii) are satisfied. Let $N_\epsilon=\{e^{i\theta}:|f(e^{i\theta})|\leqq|\alpha|+\epsilon\}$. By Corollary 2.6 and condition (ii), $\tilde{N}_\epsilon$ contains the support of the representing measure of each upper barely tangential homomorphism for every $\epsilon>0$. Thus, for every $\epsilon>0$ we have $|f|\leqq|\alpha|+\epsilon$ on each such support. Thus, $|f|\leqq|\alpha|$ on each such support. By condition (i) if $h\in\mathcal{B}^+$, then $f(h)=\alpha$. But, $f(h)$ is the integral average of values not exceeding α. Therefore, f must be identically α on the support of μ_h. By Theorem 3.1, again, f is approximately continuous from above at 1 with value α.

That condition (i) is necessary is a Lindelöf-type theorem for L^∞. Using Lemma 2.3 we state this theorem more concretely. It should be noted that because of Lemma 2.2 this theorem generalizes the usual Lindelöf theorem for H^∞: If $f\in H^\infty$ and is approximately continuous from above at 1 with value α, then f tends to α radially.

THEOREM 3.3. *Let $f\in L^\infty$. If f is approximately continuous from above at* 1 *with value α, then the upper barely tangential cluster set of f at* 1 *consists of the single point α.*

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