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PARACOMPACTNESS AND ELASTIC SPACES

HISAHIRO TAMANO[1] AND J. E. VAUGHAN

ABSTRACT. This paper gives a characterization of paracompactness, and introduces the notion of an elastic space which generalizes the concept of a stratifiable (in particular, metric) space.

1. **Introduction.** In this note we shall give a characterization of paracompactness which is formally weaker than our previous characterizations [4, Theorem 2], [5, Theorem 3], [6, Theorem 1] concerning linearly cushioned refinements. Furthermore, we shall define a new generalization of metric spaces and stratifiable spaces, called "elastic spaces," by introducing the notion of an "elastic base."

DEFINITION 1. Let $\mathcal{U}$ be a collection of subsets of a set X, and let $\mathcal{R}$ be a relation on $\mathcal{U}$ (i.e., $\mathcal{R}\subset\mathcal{U}\times\mathcal{U}$). We shall often write $U\mathcal{R}V$ instead of $(U, V)\in\mathcal{R}$. The relation $\mathcal{R}$ is said to be a *framed relation* on $\mathcal{U}$ (or a *framing* of $\mathcal{U}$) provided for every $U, V\in\mathcal{U}$, if $U\cap V\neq\emptyset$, then $U\mathcal{R}V$ or $V\mathcal{R}U$. We say $\mathcal{R}$ is a *well-framed relation* on $\mathcal{U}$ provided $\mathcal{R}$ is a framing of $\mathcal{U}$ and for every $x\in X$, there exists an $\mathcal{R}$-smallest $U_x\in\mathcal{U}$ containing x (i.e., if $x\in U$, $U\in\mathcal{U}$, and $U\neq U_x$, then $(U, U_x)\notin\mathcal{R}$).

DEFINITION 2. A collection $\mathcal{U}$ is said to be *framed in a collection* $\mathcal{V}$ *with frame map* $f:\mathcal{U}\to\mathcal{V}$ provided there exists a framed relation $\mathcal{R}$ on $\mathcal{U}$ such that for every subcollection $\mathcal{U}'\subset\mathcal{U}$ which has an $\mathcal{R}$-upper bound (i.e., there exists $U\in\mathcal{U}$ so that $U'\mathcal{R}U$ for every $U'\in\mathcal{U}'$) we have $\mathrm{cl}(\bigcup\mathcal{U}')\subset\bigcup f(\mathcal{U}')$. If in addition $\mathcal{R}$ is a well-framed relation on $\mathcal{U}$, we say that $\mathcal{U}$ is *well-framed* in $\mathcal{V}$. Finally, if $\mathcal{U}$ is framed in $\mathcal{V}$ and $\mathcal{R}$ is also a transitive relation, then $\mathcal{U}$ is called *elastic* in $\mathcal{V}$, or an *elastic refinement* of $\mathcal{V}$ when $\mathcal{U}$ and $\mathcal{V}$ are covers of X.

THEOREM 1. *Let X be a regular space. A necessary and sufficient condition that X be paracompact is that every open cover of X have an open elastic refinement.*

2. **Proof of Theorem 1.** The proof follows from the next two lemmas.

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[1] This note is a revision of a manuscript of H. Tamano which was found by J. Nagata after the death of Professor Tamano.

LEMMA 1. *Let X be a regular space. A necessary and sufficient condition for X to be paracompact is for every open cover of X to have an open refinement which is well-framed in it.*

PROOF. We shall prove the sufficiency. The proof is similar to that of Theorem 1 in [6]. Let $\mathcal{V}$ be an open cover of X. Let $\mathcal{U}$ be an open refinement of $\mathcal{V}$ which is well-framed in $\mathcal{V}$ with respect to the well-framed relation $\mathcal{R}$ on $\mathcal{U}$ and frame map $f:\mathcal{U}\to\mathcal{V}$. Let $H_U = U - \bigcup\{U'\in\mathcal{U}: U'\mathcal{R}U \text{ and } U'\neq U\}$, and $\mathcal{H}=\{H_U: U\in\mathcal{U}\}$. We now show that $\mathcal{H}$ is a cushioned refinement of $\mathcal{V}$ with cushion map $g:\mathcal{H}\to\mathcal{V}$ defined by $g(H_U)=f(U)$ (these terms are defined in [6]) and conclude that X is paracompact by [3, Theorem 1.1, p. 309]. It is easy to see that $\mathcal{H}$ is a cover of X since $\mathcal{R}$ is well-framed. It remains to show that $\mathcal{H}$ is cushioned in $\mathcal{V}$. Let $\mathcal{H}'\subset\mathcal{H}$, and suppose $x\notin\bigcup g(\mathcal{H}')$. Let U_x be an $\mathcal{R}$-smallest element of $\mathcal{U}$ containing x. Clearly U_x is an open neighborhood of x missing H_U for all $U\neq U_x$ such that $U_x\mathcal{R}U$. Further, $\mathcal{U}'=\{U\in\mathcal{U}: U\mathcal{R}U_x \text{ and } H_U\in\mathcal{H}'\}$ has an $\mathcal{R}$-upper bound. Hence $\mathrm{cl}(\bigcup\mathcal{U}')\subset\bigcup f(\mathcal{U}')\subset\bigcup g(\mathcal{H}')$ because $\mathcal{U}$ is framed in $\mathcal{V}$. Therefore, there exists an open neighborhood N of x missing $\mathrm{cl}(\bigcup\mathcal{U}')$. Finally, if U is not $\mathcal{R}$-related to U_x, then $U\cap U_x=\emptyset$ since $\mathcal{R}$ is a framing of $\mathcal{U}$. Thus, $U_x\cap N$ is an open neighborhood of x missing $\bigcup\mathcal{H}'$, and we have $\mathrm{cl}(\bigcup\mathcal{H}')\subset\bigcup g(\mathcal{H}')$. This completes the proof.

LEMMA 2. *Let $\mathcal{U}$ be a cover of a set X, and let $\leqq$ be a transitive relation which is a framing of $\mathcal{U}$. Then there exists a well-framed relation $\mathcal{R}$ on $\mathcal{U}$ so that every subset of $\mathcal{U}$ with an $\mathcal{R}$-upper bound has a $\leqq$-upper bound.*

PROOF.[2] Let $\leqq^*$ be a well-order on an index set for $\mathcal{U}$ such that $\mathcal{U}=\{U_0, U_1, \cdots, U_\alpha, \cdots : \alpha <^* \eta\}$. Let $\Lambda[\alpha]=\{U\in\mathcal{U}: U\leqq U_\alpha\}$ for all $\alpha<^*\eta$. Well-order $\Lambda[0]$ in any manner and denote the well-order by $\leqq\leqq_0$. Suppose a reflexive and antisymmetric relation $\leqq\leqq_\beta$ has been defined on $\bigcup\{\Lambda[\gamma]:\gamma\leqq^*\beta\}$ for all $\beta<^*\alpha$ in such a way that $\leqq\leqq_\beta$ is an extension of $\leqq\leqq_\delta$ whenever $\delta\leqq^*\beta$. Put $\Lambda'[\alpha]=\Lambda[\alpha]-\bigcup\{\Lambda[\beta]:\beta<^*\alpha\}$; well-order $\Lambda'[\alpha]$ and denote the order by $\leqq_\alpha$. Also define $U\leqq_\alpha U'$ if $U'\in\Lambda'[\alpha]$ and $U\in\Lambda[\alpha]\cap(\bigcup\{\Lambda[\beta]:\beta<^*\alpha\})$. Let $\leqq\leqq_\alpha$ be the relation on $\bigcup\{\Lambda[\beta]:\beta\leqq^*\alpha\}$ generated by $\{\leqq\leqq_\beta:\beta<^*\alpha\}$ and $\leqq_\alpha$. Finally, let $\mathcal{R}$ be the reflexive and antisymmetric relation generated by $\{\leqq\leqq_\alpha:\alpha<^*\eta\}$.

First we shall show that $\mathcal{R}$ is a framing of $\mathcal{U}$. If $U, U'\in\mathcal{U}$ and $U\cap U'\neq\emptyset$, then either $U\leqq U'$ or $U'\leqq U$ since $\leqq$ is a framing of $\mathcal{U}$.

[2] The authors would like to thank E. Michael for some helpful suggestions concerning this result.

Suppose that $U \leqq U'$. Let α_0 be the first index such that $U' \in \Lambda[\alpha_0]$. Since $\leqq$ is transitive, we have $U \in \Lambda[\alpha_0]$, and thus by the definition of $\mathcal{R}$ we know $U' \mathcal{R} U$ or $U \mathcal{R} U'$.

Next we show that if $\mathcal{U}$ has an $\mathcal{R}$-upper bound, then $\mathcal{U}$ has a $\leqq$-upper bound. To do this we first note that if $U \mathcal{R} U'$, then for the first index α such that $U, U' \in \bigcup \{\Lambda[\beta] : \beta \leqq {}^* \alpha\}$ we have $U \leqq \leqq_\alpha U'$, $U' \in \Lambda'[\alpha]$, and $U \in \Lambda[\alpha]$. To see this, let γ be the first index such that $U \leqq \leqq_\gamma U'$. Since $\leqq \leqq_\gamma$ is a relation on $\bigcup \{\Lambda[\beta] : \beta \leqq {}^* \gamma\}$ we know $\alpha \leqq {}^* \gamma$. If $\alpha < {}^* \gamma$, then $U, U' \notin \Lambda'[\gamma]$ so U, U' are not related by $\leqq_\gamma$. By the definition of $\leqq \leqq_\gamma$, we must have $U \leqq \leqq_\beta U'$ for some $\beta < {}^* \gamma$, but this contradicts the definition of γ. Thus, $\alpha = \gamma$, and $U \leqq \leqq_\alpha U'$. Further, not both of U and U' are in $\bigcup \{\Lambda[\beta] : \beta < {}^* \delta\}$ for any $\delta < {}^* \alpha$, so U and U' are not related by $\leqq \leqq_\delta$ for any $\delta < {}^* \alpha$. Thus, $U \leqq_\alpha U'$, from which it follows that $U' \in \Lambda'[\alpha]$ and $U \in \Lambda[\alpha]$. Now suppose $\mathcal{U}'$ is a subcollection of $\mathcal{U}$ which has an $\mathcal{R}$-upper bound. Let U' be an $\mathcal{R}$-upper bound of $\mathcal{U}'$. Let α_0 be the first index such that $U' \in \Lambda[\alpha_0]$. We now show that $\mathcal{U}'$ has U_{α_0} for $\leqq$-upper bound. Let $U \in \mathcal{U}'$ and let α_1 be the first index such that $U \in \Lambda[\alpha_1]$. If $\alpha_1 \leqq {}^* \alpha_0$, then α_0 is the first index such that $U, U' \in \bigcup \{\Lambda[\beta] : \beta \leqq {}^* \alpha_0\}$. Hence $U \mathcal{R} U'$ implies $U, U' \in \Lambda[\alpha_0]$ as noted above. In particular $U \leqq U_{\alpha_0}$ by definition of $\Lambda[\alpha_0]$. If $\alpha_0 < {}^* \alpha_1$ then α_1 is the first index such that $U, U' \in \bigcup \{\Lambda[\beta] : \beta \leqq {}^* \alpha_1\}$. Hence $U \mathcal{R} U'$ implies $U' \in \Lambda'[\alpha_1]$, but this contradicts the fact that $U' \in \Lambda[\alpha_0]$.

Finally we show that every nonempty subset $\mathcal{U}'$ of $\mathcal{U}$ has an $\mathcal{R}$-smallest element. Let α be the first index such that $\mathcal{U}' \cap \Lambda[\alpha] \neq \emptyset$. Then $\mathcal{U}' \cap \Lambda[\alpha] \subset \Lambda'[\alpha]$. Since $(\Lambda'[\alpha], \leqq_\alpha)$ is a well-ordered subset of $\mathcal{U}$, there exists a $\leqq_\alpha$-first element of $\mathcal{U}' \cap \Lambda[\alpha]$ which is an $\mathcal{R}$-smallest element of $\mathcal{U}'$. Thus $\mathcal{R}$ is well-framed, and this completes the proof.

Proof of Theorem 1. We need only prove the sufficiency. Let $\mathcal{U}$ be an open cover of X, and let $\mathcal{W}$ be an open elastic refinement of $\mathcal{U}$. By Lemma 2, it is easy to see that there is a well-framed relation on $\mathcal{W}$, and that $\mathcal{W}$ is well-framed in $\mathcal{U}$. Hence X is paracompact by Lemma 1.

3. **Elastic spaces.** According to J. G. Ceder [2], a collection $\boldsymbol{P}$ of ordered pairs $P = (P_1, P_2)$ of subsets of a space X is called a *pair base* for X provided that P_1 is open for all $P \in \boldsymbol{P}$ and that for every $x \in X$ and for every open set U containing x, there exists a $P \in \boldsymbol{P}$ such that $x \in P_1 \subset P_2 \subset U$. Further, he called a T_1-space an M_3-space (renamed *stratifiable space* by C. J. R. Borges [1]) provided it has a σ-cushioned pair base $\boldsymbol{P}$. A pair base $\boldsymbol{P}$ is said to be *σ-cushioned* provided $\boldsymbol{P}$

$=\bigcup_{n=1}^{\infty} \boldsymbol{P}_n$, and for every n and every $\boldsymbol{P}_n' \subset \boldsymbol{P}_n$ we have

$$\mathrm{cl}(\bigcup\{P_1 : P \in \boldsymbol{P}_n'\}) \subset \bigcup\{P_2 : P \in \boldsymbol{P}_n'\}.$$

DEFINITION 3. A pair base $\boldsymbol{P}$ for a space X is said to be an *elastic base* if there is a framing of $\boldsymbol{P}_1 = \{P_1 : P = (P_1, P_2) \in P\}$ such that $\boldsymbol{P}_1$ is elastic in $\boldsymbol{P}_2 = \{P_2 : P = (P_1, P_2) \in \boldsymbol{P}\}$ with respect to the map $f(P_1) = P_2$. A T_1-space with an elastic base is called an *elastic space.*

THEOREM 2. *Every subspace of an elastic space is an elastic space. Every metrizable space, and more generally every stratifiable space, is an elastic space. Every elastic space is paracompact.*

PROOF. The first statement is obvious. Let X be a stratifiable space with a σ-cushioned pair base $\boldsymbol{P} = \bigcup_{n=1}^{\infty} \boldsymbol{P}_n$. We may assume that $\{\boldsymbol{P}_n : n = 1, 2, \cdots\}$ is a partition of $\boldsymbol{P}$. Let $\leqq_n$ be a well-order on $\boldsymbol{P}_n$ for each n, and define a well-order $\leqq$ on $\boldsymbol{P}$ as follows: For $P, P' \in \boldsymbol{P}$ we say $P \leqq P'$ if and only if either (1) P, P' are in the same $\boldsymbol{P}_n$ and $P \leqq_n P'$, or (2) $P \in \boldsymbol{P}_n$, $P' \in \boldsymbol{P}_m$, and $n < m$. Then $\boldsymbol{P}$ obviously is an elastic base. Since an elastic space is regular, it follows from Theorem 1 that every elastic space is paracompact.

EXAMPLE. (An elastic space which is not a stratifiable space.) Let $X = [0, \Omega]$ be the set of ordinals less than or equal to the first uncountable ordinal. Let the topology on X be the weakest topology stronger than the order topology for which every point is isolated except Ω. Construct an elastic base for X as follows. Let $U_\alpha = (\alpha, \Omega]$ for all $\alpha < \Omega$, and let $\boldsymbol{P}' = \{(U_\alpha, U_\alpha) : \alpha \in [0, \Omega)\}$ and order $\boldsymbol{P}'$ by the usual order on the index set $[0, \Omega)$. Let $W_\alpha = \{\alpha\}$ for all $\alpha < \Omega$, and let $\boldsymbol{P}'' = \{(W_\alpha, W_\alpha) : \alpha \in [0, \Omega)\}$ and order $\boldsymbol{P}''$ by the usual order on the index set $[0, \Omega)$. Finally, set $\boldsymbol{P} = \boldsymbol{P}' \cup \boldsymbol{P}''$ and order $\boldsymbol{P}$ so that every element of $\boldsymbol{P}'$ precedes every element of $\boldsymbol{P}''$. Then $\boldsymbol{P}$ is an elastic base for X, so X is an elastic space. Clearly, X is not stratifiable because the closed set $\{\Omega\}$ is not a G_δ in X (see [**2**, Theorem 2.2, p. 106]).

CONJECTURE.[3] *Every closed continuous image of an elastic space is an elastic space.*

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[3] Tamano called this statement a theorem in his manuscript, but he did not give a complete proof; so it is stated here as a conjecture.

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Texas Christian University, Fort Worth, Texas 76129

University of North Carolina, Chapel Hill, North Carolina 27514