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THE POWER PROBLEM FOR GROUPS WITH ONE DEFINING RELATOR

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ABSTRACT. It is proved that if G is a group with one defining relator, then the generalized word problem is solvable for every cyclic subgroup of G. This result enables the solution of the word problem for groups with one defining relator to be extended to a wider class of groups.

1. **Introduction.** Let G be a group given by a presentation $\langle S; D\rangle$, and let H be a subgroup of G, generated by a set A of words in the elements of S. The generalized word problem (GWP) for H in G is the algorithmic problem of deciding whether or not an arbitrary word $U\in G$ is an element of H. If the GWP is solvable for every cyclic subgroup of G, then G is said to have solvable power problem. The object of this note is to prove the following

THEOREM. *Let G have presentation $\langle t, b, c, \cdots ; R(t, b, c, \cdots)\rangle$. Then the power problem is solvable for G.*

The order problem for a group $G=\langle S; D\rangle$ is the algorithmic problem of deciding the order of an arbitrary word $U\in G$. A solution to this problem for groups with a single defining relator is given by the results of §4.4 of [**2**]. It follows that Theorem 6 of [**3**] (see also Theorem 5 of [**1**]) can be applied to extend the solution of the word problem for groups with a single defining relator to a wider class of groups. As a simple example of this, we have the following generalization of Corollary 4.14.1 of [**2**].

COROLLARY. *Let G_1 and G_2 have presentations*

$$\langle a_1, \cdots, a_m; R(a_1, \cdots, a_m)\rangle \quad \textit{and} \quad \langle b_1, \cdots, b_n; S(b_1, \cdots, b_n)\rangle$$

respectively, and let $U(a_1, \cdots, a_m)$, $V(b_1, \cdots, b_n)$ be elements of G_1, G_2 respectively, such that the orders of these elements are equal. Then the order problem and the power problem are solvable for the group

$$G = \langle a_1, \cdots, a_m, b_1, \cdots, b_n; R, S, U = V\rangle.$$

2. **Proof of the theorem.** We shall make use of the following generalization of Lemma 2 of [**1**]. The proof requires only trivial

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modification of that of Theorem 5 of [1], and so is omitted.

LEMMA. *Let the groups G_0 and G_1 have presentations*

$$\langle a_1, a_2, \cdots, b_1, b_2, \cdots; R_1, R_2, \cdots \rangle$$

and

$$\langle a_1, a_2, \cdots, c_1, c_2, \cdots; S_1, S_2, \cdots \rangle,$$

respectively, and suppose that the following conditions are satisfied:

(a) *The power problem is solvable for G_0 and G_1.*

(b) *The subgroups H_i of G_i $(i=0, 1)$ generated by the corresponding elements $a_1, a_2, \cdots$ are isomorphic under the identity mapping.*

(c) *The GWP for H_i in G_i $(i=0, 1)$ is solvable.*

Then the power problem is solvable for

$$G = \langle a_1, a_2, \cdots, b_1, b_2, \cdots, c_1, c_2, \cdots; R_1, R_2, \cdots, S_1, S_2, \cdots \rangle,$$

the free product of G_0 and G_1 amalgamating H_0 with H_1.

We prove the theorem by induction on the length of the relator R. The method of proof is the one used repeatedly in §4.4 of [2], so we have omitted many of the details.

We can suppose that R as written is cyclically reduced. If R involves only one generator, it is easy to see that the result holds. Thus we assume that R involves at least two generators, say t and b, and that the result holds for all groups with one defining relator of length less than that of R.

Case 1. R has zero exponent sum on some generator; say $\sigma_t(R)=0$.

Let U and V be elements of G. We show that we can decide whether or not there exists an integer n such that $U=V^n$. Using the solution of the order problem for G, it is easy to dispose of the case when either U or V has finite order; thus we assume that U and V have infinite order.

Now if $U=V^n$ for some integer n (which must be nonzero), then $UV^{-n}\in N$, the normal subgroup of G generated by $b, c, \cdots$, and so $\sigma_t(UV^{-n})=0$. Thus, putting $\lambda=\sigma_t(U)$ and $\eta=\sigma_t(V)$, we must have $\lambda-n\eta=0$.

Suppose that $\lambda\neq 0$. Then, if $U=V^n$, we have η divides λ and $n=\lambda/\eta$. Thus in this case there is at most one value of n to test.

Thus we can assume that $\lambda=0$. If $\eta\neq 0$, then, since $n\neq 0$, we cannot have $\lambda-n\eta=0$. Thus we can assume also that $\eta=0$. In other words, we can assume that both U and V are elements of N.

We now show that N has solvable power problem. We have

$$N = \langle \cdots, b_{-1}, b_0, b_1, \cdots, c_{-1}, c_0, c_1, \cdots; \cdots, P_{-1}, P_0, P_1, \cdots \rangle,$$

where, for $k=0, \pm 1, \pm 2, \cdots, b_k, c_k, \cdots$ denote the elements $t^kbt^{-k}, t^kct^{-k}, \cdots$ respectively, and P_k is the element t^kRt^{-k} rewritten in terms of these generators.

Now the subgroup N_i of N generated by

$$\cdots, c_{-1}, c_0, c_1, \cdots, b_{\mu+i}, \cdots, b_{M+i},$$

where μ is the minimum subscript on b involved in P_0, and M is the maximum subscript on b involved in P_0, has presentation

$$N_i = \langle \cdots, c_{-1}, c_0, c_1, \cdots, b_{\mu+i}, \cdots, b_{M+i}; P_i \rangle.$$

Thus N_i is a group with one defining relator P_i. Moreover, the length of P_i is less than that of R, so that, by the inductive hypothesis, the power problem is solvable for N_i.

Now, as in the proof of Theorem 4.14 of [**2**], we can describe N as the union of a chain of groups

$$Q_1 = N_0 \subset Q_2 \subset \cdots \subset Q_s \subset Q_{s+1} \subset \cdots.$$

We prove, by induction on s, that each Q_s has solvable power problem; it will then follow that N has solvable power problem.

We suppose that the power problem is solvable for Q_s. Now Q_{s+1} is the free product of Q_s and some N_p, with the subgroup K of N_p generated by all the generators of N_p except some b_k amalgamated under the identity mapping. Denote this set of generators by A. Then A is a subset of the generators of some N_q whose generators are among the generators of Q_s. Now the GWP for K in N_p is solvable, as is the GWP for K in N_q, by Theorem 4.14 of [**2**]; moreover, in the proof of that theorem, it is shown that if the generators of N_q are among the generators of Q_s, then the GWP for N_q in Q_s is solvable. It follows that the GWP for K in Q_s is solvable. Thus we can apply the lemma, to deduce that the power problem is solvable for Q_{s+1}. Hence the power problem is solvable for each Q_s, and so is also solvable for N.

Case 2. All the generators in R have nonzero exponent sums.

Put $\alpha=\sigma_t(R)$ and $\beta=\sigma_b(R)$. Then G is (effectively) embedded in the group G_1 with presentation

$$G_1 = \langle x, y, c, \cdots; R(yx^{-\beta}, x^\alpha, c, \cdots) \rangle.$$

Thus the power problem is solvable for G if it is solvable for G_1. But the exponent sum of x in $R(yx^{-\beta}, x^\alpha, c, \cdots)$ is zero, and when this relator is rewritten in terms of the (usual) generators of the normal subgroup of G_1 generated by y, c, $\cdots$, the relator obtained has length less that that of $R(t, b, c, \cdots)$. Thus, using the same argument as in Case 1, we see that the power problem is solvable for G_1. This proves the theorem.

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