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DIMENSION-THEORETIC PROPERTIES OF COMPLETIONS

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ABSTRACT. In this paper we extend some previous work from situations involving countable collections of subsets to those concerning locally finite collections. An example of the results obtained here is a theorem which asserts that corresponding to any locally finite collection of finite-dimensional closed subsets of a metric space X there exists a completion of X in which taking the closure of any member of the given collection does not raise dimension. The basic technique employed in each of the proofs is similar; a topologically equivalent metric is introduced (one which is strongly dependent upon the given locally finite collection), and the desired completion is then taken with respect to this new metric.

The author has previously shown [4] that for every countable collection $\{X_i\}$ of finite-dimensional closed subsets of a metric space X, there exists a completion of X in which the closure of each X_i has the same dimension as X_i (by the term completion of X we mean a complete metric space in which X can be topologically embedded as a dense subset; the word dimension will denote the covering dimension of Lebesgue, cf. [2]). One might ask whether an analogous result would hold in the case of a locally finite, rather than a countable, collection of subsets; the answer to this question is yes, as we shall prove in the following.

THEOREM 1. *Let $\{F_\lambda:\lambda\in\Lambda\}$ be a locally finite collection of finite-dimensional closed subsets of a metric space X. Then there exists a completion X^* of X such that*

$$\dim(\mathrm{cl}_{X^*}F_\lambda) = \dim F_\lambda \qquad \textit{for all } \lambda \in \Lambda.$$

Moreover, if X is separable then X^* is a metric compactification of X.

Theorem 1 will follow from the following more general theorem:

THEOREM 2. *Let X be a (respectively, separable) metric space, Y a complete (resp., compact) metric space, $\{F_\lambda:\lambda\in\Lambda\}$ a closure-preserving collection of finite-dimensional closed subsets of X, and*

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$\{f_i : i=1, 2, \cdots\}$ a sequence of continuous mappings from X into Y. Then there exists a completion (resp., metric compactification) X^* of X to which each f_i can be continuously extended and such that

$$\dim(\mathrm{cl}_{X^*}F_\lambda) = \dim F_\lambda \qquad \textit{for each } \lambda \in \Lambda.$$

PROOF. For each $k=0, 1, 2, \cdots$, we define

$$X_k = \bigcup\{F_\lambda : \lambda \in \Lambda \text{ and } \dim F_\lambda = k\}.$$

By hypothesis, each F_λ is a subset of some X_k, and each X_k is closed; also, by a sum theorem of Nagami [**1**, Theorem 1], $\dim X_k = k$ for each nonvoid X_k. Let X^* be a completion (resp., metric compactification) of X constructed in a previous work [**5**, Theorem 3 (resp., Theorem 4)]. Each f_i can be continuously extended to X^*; also, for any $\lambda \in \Lambda$, we let $k(\lambda) = \dim F_\lambda$, in which case

$$\begin{aligned} k(\lambda) &= \dim F_\lambda \leqq \dim(\mathrm{cl}_{X^*}F_\lambda) \\ &\leqq \dim(\mathrm{cl}_{X^*}X_{k(\lambda)}) = \dim X_{k(\lambda)} = k(\lambda) \end{aligned}$$

by the Monotone Theorem [**2**, Theorem II.3] and the previously cited theorem of the author. This completes this proof.

Similar techniques can be applied to other problems as well. In the following we shall introduce an equivalent metric (into a metric space) which has certain dimension-theoretic properties on each member of a given closure-preserving family of subspaces. The following definitions will be needed.

DEFINITIONS. Let (X, ρ) be a metric space, and let $Y \subset X$, $\dim Y = n$, and ρ_Y be the induced metric on Y. Define $S_\alpha(x|Y) = \{y \in Y : \rho_Y(x, y) < \alpha\}$ for any $x \in y$ and any $\alpha > 0$.

We say that ρ *has Property A on* Y if and only if there exists $\delta > 0$ such that for every positive $\epsilon < \delta$ and every $x \in Y$,

$$\rho_Y(S_{\epsilon/2}(x|Y), y_i) < \epsilon \qquad (i = 1, \cdots, n+2)$$

imply

$$\rho_Y(y_i, y_j) < \epsilon \qquad \text{for some } i, j \text{ with } i \neq j.$$

We say ρ *is dimension-lowering on* Y if and only if for every $\epsilon > 0$ and every $x \in X$, $\dim[Y \cap B(S_\epsilon(x))] < n$.

We say ρ *is strongly dimension-lowering on* Y if and only if for every $\epsilon > 0$ and every subset D of X, $\dim[Y \cap B(S_\epsilon(D))] < n$.

THEOREM 3. *Let X be a (resp., separable) metric space with bounded (resp., totally bounded) metric d, and $\{F_\lambda : \lambda \in \Lambda\}$ be a closure-preserving collection of nonvoid finite-dimensional closed subsets of X. Then there* *exists a topologically equivalent (resp., totally bounded) metric ρ for X such that*

(i) *ρ has Property A on each F_λ,*

(ii) *ρ is dimension-lowering on each F_λ,*

(iii) *for all $\epsilon>0$, $\{S_\epsilon(x):x\in X\}$ is closure-preserving (resp., finite), and*

(iv) *$d(x, y)\leqq\rho(x, y)\leqq d(X)$ for all $x, y\in X$.*

PROOF. For each $k=0, 1, 2, \cdots$, we define

$$X_k = \bigcup\{F_\lambda:\lambda\in\Lambda \text{ and } \dim F_\lambda = k\}.$$

As in the proof of Theorem 2, each nonvoid X_k is closed and k-dimensional, so by an earlier theorem of the author [5, Theorem 1 (resp., Theorem 2)] we can introduce an equivalent metric ρ which satisfies (iii) and (iv) as well as

(i′) ρ has Property A on each X_k, and

(ii′) ρ is a dimension-lowering on each nonvoid X_k.

Now for any $\lambda\in\Lambda$ we let $k(\lambda)=\dim F_\lambda$. Since

$$\dim X_{k(\lambda)}=k(\lambda)=\dim F_\lambda,$$

it is easy to see that conditions (i′) and (ii′) imply (i) and (ii), respectively, and the theorem follows.

Since any metric can be replaced by a topologically equivalent bounded metric (a totally bounded metric in the separable case), Theorem 3 can be restated as follows:

THEOREM 4. *If, in Theorem 3, X is any (resp., separable) metric space, then there exists a topologically equivalent (resp., totally bounded) metric ρ for X which satisfies conditions* (i), (ii), *and* (iii).

We close with an application of Theorem 4.

THEOREM 5. *Let $\mathfrak{F}$ be a closure-preserving collection of nonvoid finite-dimensional closed subsets of a (resp., separable) metric space X. Then there exists a topologically equivalent (resp., totally bounded) metric for X which is strongly dimension-lowering on each member of $\mathfrak{F}$.*

PROOF. We let ρ be the metric introduced in Theorem 4, and let $F\in\mathfrak{F}$, $k=\dim F$, $D\subset X$, and $\epsilon>0$. To prove that $F\cap B(S_\epsilon(D))$ has dimension $<k$ we first show that it is covered by the closure-preserving collection $\mathfrak{H}=\{F\cap B(S_\epsilon(D))\cap B(S_\epsilon(p)):p\in D\}$. To see that $\mathfrak{H}$ is a cover, we note that

$$
\begin{aligned}
F \cap B(S_\epsilon(D)) &= F \cap B(S_\epsilon(D)) \cap \left(\overline{\bigcup_{p\in D} S_\epsilon(p)} - \bigcup_{p\in D} S_\epsilon(p) \right) \\
&= F \cap B(S_\epsilon(D)) \cap \left(\bigcup_{p\in D} \overline{S_\epsilon(p)} - \bigcup_{p\in D} S_\epsilon(p) \right) \\
&\qquad\qquad\qquad\qquad \text{(by (iii) of Theorem 4)} \\
&\subset F \cap B(S_\epsilon(D)) \cap \bigcup_{p\in D} (\overline{S_\epsilon(p)} - S_\epsilon(p)) \\
&= F \cap B(S_\epsilon(D)) \cap \bigcup_{p\in D} B(S_\epsilon(p)) \\
&= \bigcup_{p\in D} (F \cap B(S_\epsilon(D)) \cap B(S_\epsilon(p))).
\end{aligned}
$$

Also, for any $E \subset D$,

$$
\begin{aligned}
&\bigcup_{p\in E} (F \cap B(S_\epsilon(D)) \cap B(S_\epsilon(p))) \\
&\qquad = F \cap \bigcup_{p\in E} ((\overline{S_\epsilon(p)} - S_\epsilon(p)) \cap B(S_\epsilon(D))) \\
&\qquad = F \cap \bigcup_{p\in E} ((\overline{S_\epsilon(p)} \cap (X - S_\epsilon(p)) \cap B(S_\epsilon(D))) \\
&\qquad = F \cap \bigcup_{p\in E} ((\overline{S_\epsilon(p)} \cap B(S_\epsilon(D))) \\
&\qquad\qquad \text{(since } B(S_\epsilon(D)) \subset X - S_\epsilon(p) \text{ for all } p \in E) \\
&\qquad = F \cap B(S_\epsilon(D)) \cap \bigcup_{p\in E} \overline{S_\epsilon(p)},
\end{aligned}
$$

which is closed by condition (iii) of Theorem 4, so $\mathfrak{H}$ is closure-preserving. By (ii) of Theorem 4,

$$\dim(F \cap B(S_\epsilon(p)) \leqq k - 1 \qquad \text{for each } p \in D,$$

so by the Monotone Theorem each member of $\mathfrak{H}$ has dimension $\leqq k-1$. Since $\mathfrak{H}$ is a closure-preserving closed cover consisting of sets of dimension $\leqq k-1$, we again apply the Sum Theorem of Nagami to infer that

$$\dim(F \cap B(S_\epsilon(D))) \leqq k - 1.$$

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