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KLEIN BOTTLES IN CIRCLE BUNDLES

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ABSTRACT. We prove that the Klein bottle embeds in the total space E of an orientable S^1-bundle over an orientable 2-manifold M if and only if $M=S^2$ and $E=S^1\times S^2$ or the lens space $L(4, 1)$.

In this note we apply results of [1] to generalize a result given there concerning the embedding of the Klein bottle.

PROPOSITION. *The Klein bottle embeds in the total space E of an orientable S^1-bundle over an orientable 2-manifold M if and only if $M=S^2$ and $E=S^1\times S^2$ or the lens space $L(4, 1)$.*

To show $M=S^2$ we use the following result of [1]:

THEOREM [1, §4.1]. *Let $i:K\rightarrow E$ be an embedding of a nonorientable $(n-1)$-manifold K in an orientable n-manifold E. Suppose that $\alpha\in\pi_1(K)$ reverses orientation. Then for $\beta\in\pi_1(E)$, $\beta^{-1}i_\#(\alpha)\beta\in i_\#(\pi_1(K))$ implies $\beta\in i_\#(\pi_1(K))$.*

Assume $M\neq S^2$, so $\pi_2(M)=0$. In the exact sequence of the fibration

$$\cdots\rightarrow 0\rightarrow \pi_1(S^1)\rightarrow \pi_1(E)\xrightarrow{p_\#}\pi_1(M)\rightarrow 0$$

the generator of $\pi_1(S^1)$ is mapped to an element g in the center of $\pi_1(E)$. (Since E is trivial over the 1-skeleton of M, the inverse image of any circle in M is a torus in E. Hence g commutes with a basis for $\pi_1(E)$.) By the theorem g is in the image of $i_\#$. Let $\pi_1(K) =\{\alpha,\ \beta:\alpha\beta\alpha^{-1}=\beta^{-1}\}$; α is the orientation reversing element. Then $g=i_\#(\alpha^j\beta^k)$ for some integers j, k. Since $\alpha\beta^k\alpha^{-1}=\beta^{-k}$, we have

$$gi_\#(\alpha^{-j+1})gi_\#(\alpha^{-j-1}) = i_\#(\alpha^j\beta^k\alpha^{-j+1}\alpha^j\beta^k\alpha^{-j-1}) = 1.$$

Therefore $g^2=i_\#(\alpha^{2j})$. $p_\#(g)=0$ and $\pi_1(M)$ is torsion free, so $p_\#i_\#(\alpha)=0$. Therefore $i_\#(\alpha)=g^m$ and is in the center of $\pi_1(E)$. But then by the theorem $i_\#$ is onto. Thus $p_\#i_\#(\beta)$ generates $\pi_1(M)$ which contradicts $M\neq S^2$.

To complete the proof of the proposition recall that the total space E of an orientable S^1-bundle over S^2 is the lens space $L(k, 1)$ or $S^1\times S^2$ (the case $k=0$). By [1, §6] a nonorientable surface of genus g embeds

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in $L(k, 1)$ if and only if k is even, $g \equiv k/2 \pmod 2$, and $g \geqq k/2$. Thus the Klein bottle, which has genus 2, embeds only in $L(4, 1)$ and $S^1 \times S^2$.

If $S^1 \times S^2$ is pictured as a family of 2-spheres parameterized by θ, $0 \leqq \theta < 2\pi$, then the surface swept out by a meridian rotated about the poles by $\theta/2$ is a Klein bottle.

In the x, y-plane let S be the square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. $L(4, 1)$ is obtained from the suspension from $(0, 0, \pm 1)$ of S in $\boldsymbol{R}^3$ by identifying certain points of the boundary, cf. [**2**, p. 223]. The surface $z = xy$ gives an embedding of the Klein bottle.

References

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