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SOME CONSEQUENCES OF dim proj $\Omega(A)<\infty$

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ABSTRACT. Let X be an affine variety over a field k and x a point on X. We are interested in relating the properties of $\Omega(X)_x$, the Kähler module of differentials of x, with geometric properties of X at x. Lipman has given necessary and sufficient conditions for $\Omega(X)_x$ to be respectively torsion free and reflexive in the case where X is locally a complete intersection at x. We give a generalization of these results for the case where the projective dimension (dim proj) of $\Omega(X)_x$ is finite.

1. Throughout $X=\operatorname{Spec} A$ will be an affine variety over a perfect ground field k; that is A is a finitely generated k-algebra without nonzero nilpotents. x will denote a point on X with local ring R.

Recall that for any k-algebra S there is a "universal k-derivation" $d: S\rightarrow\Omega(S)$, where $\Omega(S)$ is the S module of k-differentials of S, so that for any S module C the homomorphism $\operatorname{Hom}_S(\Omega(S), C)\rightarrow\operatorname{Der}_k(S, C)$, $f\rightarrow fd$, is an isomorphism. $\Omega(S)$ may be constructed explicitly as I/I^2 where I is the kernel of the multiplication $S\otimes S\rightarrow S$, and $d(s)=s\otimes 1-1\otimes s$.

We will use the notation of Buchsbaum [2] with the exception that we write dim proj M for hd M. We make extensive use of the theorem: dim proj $M+\operatorname{codim} M=\operatorname{codim} R$ for an R module M with dim proj $M<\infty$. A proof can be found in [2].

THEOREM 1. *Let R be Macaulay and let* dim proj $\Omega(R)=h<\infty$.

(1) *If X is nonsingular in codimension h at x, then $\Omega(R)$ is torsion free.*

(2) *If X is nonsingular in codimension $h+1$ at x, then $\Omega(R)$ is reflexive.*

PROOF. (1) As R is Macaulay codim $R=\dim R$ and so codim $\Omega(R)=\operatorname{codim} R-h=\dim R-h$. But if $\mathfrak{p}\in\operatorname{Ass}\Omega(R)$, then codim $\Omega(R)\leqq\dim R/\mathfrak{p}=\dim R-\operatorname{height}\mathfrak{p}$, so height $\mathfrak{p}\leqq h$.

Now if $\mathfrak{p}\in\operatorname{Ass}\Omega(R)$, then $\mathfrak{p}R_{\mathfrak{p}}\in\operatorname{Ass}(\Omega(R_{\mathfrak{p}}))$. But as X is nonsingular in codimension h at x and ht $\mathfrak{p}\leqq h$, $\Omega(R_{\mathfrak{p}})$ is free and

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[1] My thanks to the referee for improving the exposition and providing a simpler proof for Theorem 3.

$\mathfrak{p} \in \operatorname{Ass} R_{\mathfrak{p}}$. Thus $\mathfrak{p} \in \operatorname{Ass} \Omega(R)$ implies $\mathfrak{p} \in \operatorname{Ass} R$, i.e. $\Omega(R)$ is torsion free.

(2) By (1), $\Omega(R)$ is torsion free. If $h=0$ there is nothing to do; otherwise $h+1 \geqq 2$ and X is nonsingular in codimension 2 at x. Vasconcelos [7, Theorem 1.4] gives the following criterion for reflexivity: If for every prime ideal $\mathfrak{q}$ of height 1 in R, $R_{\mathfrak{q}}$ is a Gorenstein ring and, for every prime ideal $\mathfrak{p}$ of height 2, $\mathfrak{p}R_{\mathfrak{p}}$ has depth 2, then an R module M is reflexive if and only if every R sequence of length 2 is an M sequence. Since X is nonsingular in codimension 2 at x, this applies to R.

Let (a, b) be an R sequence. Let $\mathfrak{p} \in \operatorname{Ass}(\Omega(R)/a\Omega(R))$. Since a is a regular element for $\Omega(R)$,

$$\operatorname{codim} \Omega(R)/a\Omega(R) = \operatorname{codim} \Omega(R) - 1 \leqq \dim R/\mathfrak{p} = \dim R - \operatorname{height} \mathfrak{p}.$$

But $\operatorname{codim} \Omega(R) = \dim R - h$, so height $\mathfrak{p} \leqq h+1$. As X is regular in codimension $h+1$ at x, $\Omega(R_{\mathfrak{p}})$ is free and hence $\operatorname{Ass}(R_{\mathfrak{p}}/aR_{\mathfrak{p}}) = \operatorname{Ass}(\Omega(R_{\mathfrak{p}})/a\Omega(R_{\mathfrak{p}}))$. Thus $\mathfrak{p}$ is an associated prime of R/aR which does not contain b since (a, b) is an R sequence. Since b is not in any prime ideal associated to $\Omega(R)/a\Omega(R)$, (a, b) is an $\Omega(R)$ sequence.

If R is not Macaulay, the methods used above give the following:

THEOREM 2. *Let* s = *Krull dimension* R − codim R, *and let* dim proj $\Omega(R) = h < \infty$.

(1) *If* X *is nonsingular in codimension* $h+s$ *at* x, *then* $\Omega(R)$ *is torsion free.*

(2) *If* X *is nonsingular in codimension* $h+s+1$ *at* x, *then* $\Omega(R)$ *is reflexive.*

As a partial converse to Theorem 1, we have the following:

THEOREM 3. *Suppose* dim proj $\Omega(R) < \infty$.

(1) *If* $\Omega(R)$ *is torsion free, then* X *is nonsingular in codimension* 1 *at* x.

(2) *If* $\Omega(R)$ *is reflexive, then* X *is nonsingular in codimension* 2 *at* x.

PROOF. (1) Let $\mathfrak{p}$ be a minimal prime ideal of R. Then dim proj $\Omega(R_{\mathfrak{p}})$ + codim $\Omega(R_{\mathfrak{p}})$ = codim $R_{\mathfrak{p}} \leqq 1$. Since $\Omega(R_{\mathfrak{p}})$ is torsion free, any nonzero divisor in $R_{\mathfrak{p}}$ is an $\Omega(R_{\mathfrak{p}})$ sequence of length 1; thus codim $R_{\mathfrak{p}}$ = codim $\Omega(R_{\mathfrak{p}})$, dim proj $\Omega(R_{\mathfrak{p}}) = 0$ and $R_{\mathfrak{p}}$ is regular.

(2) Let $\mathfrak{p}$ be a prime ideal of R of height 2. Since any $R_{\mathfrak{p}}$ sequence of length $\leqq 2$ is an $\Omega(R_{\mathfrak{p}})$ sequence of the same length, codim $\Omega(R_{\mathfrak{p}})$ = codim $R_{\mathfrak{p}}$. Thus dim proj $\Omega(R_{\mathfrak{p}}) = 0$ and $R_{\mathfrak{p}}$ is regular.

The above theorems generalize the following result of Lipman [4, p. 896, Proposition 8.1]: If X is locally a complete intersection at

x, then $\Omega(R)$ is torsion free if and only if X is nonsingular in codimension 1 at x, and $\Omega(R)$ is reflexive if and only if X is nonsingular in codimension 2 at x. This is the case dim proj $\Omega(R)=1$ of our theorems.

2. If R is a Macaulay ring and $h=\dim \operatorname{proj} \Omega(R)$, the proof of the first part of Theorem 1 shows that $h \geqq$ height $\mathfrak{p}$ for $\mathfrak{p}$ any associated prime ideal of $\Omega(R)$. Thus if X is nonsingular in codimension r at x, and R is Macaulay, torsion in $\Omega(R)$ implies dim proj $\Omega(R) \geqq r+1$. Alternatively if $\dim X=n$, x is an isolated singular point and R is Macaulay, then torsion in $\Omega(R)$ implies dim proj $\Omega(R) \geqq n$. We show below that in "good characteristic" the quotient of a nonsingular variety acted upon by a finite group is Macaulay. And so it follows that torsion in $\Omega(R)$ implies dim proj $\Omega(R) \geqq \dim X$, where R is the local ring of an isolated singularity on such a quotient variety.

If Spec A is a nonsingular variety over k and G is a finite group of automorphisms of Spec A, then Spec A^G is the quotient of X by G [**3**]. So we need only prove the following:

THEOREM 4. *If the characteristic of k is prime to the order of G and A is a locally Macaulay ring, then A^G is a locally Macaulay ring.*

PROOF. In the following extension and contraction (denoted respectively by $\mathfrak{a}^e$, $\mathfrak{b}^c$) will always mean extension and contraction with respect to the inclusion $A^G \to A$. By the Unmixedness Theorem, it suffices to show that an ideal of A^G of height h with h generators is unmixed, i.e. that all its associated prime ideals have the same height. Let $\mathfrak{a}$ be such an ideal in A^G. $\mathfrak{a}^e$ also has h generators. If $\mathfrak{p}$ is a prime ideal of A^G containing $\mathfrak{a}$ and $\mathfrak{q}$ is a prime ideal of A lying over $\mathfrak{p}$, then $\mathfrak{q}$ contains $\mathfrak{a}^e$. Since A is integral over A^G, height $\mathfrak{p}$ = height $\mathfrak{q}$ so height $\mathfrak{a}^e=h$. Thus $\mathfrak{a}^e$ is an ideal of A of height h with h generators. Since A is locally Macaulay $\mathfrak{a}^e$ is unmixed. But the associated prime ideals of $\mathfrak{a}^{ec}$ are some subset of the contractions of the associated prime ideals of $\mathfrak{a}^e$, so $\mathfrak{a}^{ec}$ is also unmixed. To complete the proof we must show that $\mathfrak{a}^{ec}=\mathfrak{a}$. Let $\mathfrak{a}=(a_1, \cdots, a_h)$ and $u=u_1a_1+\cdots+u_ha_h \in \mathfrak{a}^{ec}$. Since $\sigma(u)=u$ for all $\sigma \in G$, $u_1a_1+\cdots+u_ha_h = \sigma(u_1)a_1+\cdots+\sigma(u_h)a_h$ so that $u=u_1'a_1+\cdots+u_h'a_h$ where $u_i'=(1/g)\sum_{\sigma\in G}\sigma(u_i)$, g is the order of G. Since $u_i' \in A^G$, $u \in \mathfrak{a}$ and $\mathfrak{a}^{ec}=\mathfrak{a}$.

The "good characteristic" hypothesis is not irrelevant. For example let $A=k[X, Y, Z, U]$ where the characteristic of k is 2 and let $G=\langle\sigma\rangle$ be the cyclic group of order 4, where σ acts on A by $\sigma(X)=Y$, $\sigma(Y)=Z$, $\sigma(Z)=U$ and $\sigma(U)=X$. Then Bertin [**1**] has shown that A^G is not a locally Macaulay ring.

3. In [**6**] Vasconcelos proved that if dim proj $\Omega(A)<\infty$ and no

height 1 prime ideal of A annihilates an element of $\Omega(A)$, then A is integrally closed. Careful inspection of his proof reveals that he needed only dim proj$(A) < \infty$ and Spec A nonsingular in codimension one as hypotheses. Below we give a simpler proof of this result.

THEOREM 5. *Let A be a finitely generated k-domain with* dim proj $\Omega(A) < \infty$. *If* Spec A *is nonsingular in codimension* 1, *then A is integrally closed.*

PROOF. By Serre's Criterion [5, Proposition 9 III-13] A is integrally closed if and only if (1) $A_{\mathfrak{p}}$ is a discrete valuation ring for every minimal prime ideal $\mathfrak{p}$ of A and (2) no prime ideal of height $\geqq 2$ is an associated prime ideal of any principal ideal. We need only verify the second condition.

Suppose $\mathfrak{p} \in \mathrm{Ass}(A/aA)$ for some $a \in A$. Since

$$\dim \operatorname{proj} \Omega(A_{\mathfrak{p}}) + \operatorname{codim} \Omega(A_{\mathfrak{p}}) = \operatorname{codim} A_{\mathfrak{p}} = 1,$$

dim proj $\Omega(A_{\mathfrak{p}}) \leqq 1$. Thus by Theorem 1, $\Omega(A_{\mathfrak{p}})$ is torsion free. But then codim $\Omega(A_{\mathfrak{p}}) = 1$, $\Omega(A_{\mathfrak{p}})$ is projective and $A_{\mathfrak{p}}$ is a regular local ring of dimension $\leqq 1$. In particular height $\mathfrak{p} \leqq 1$.

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