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COMMUTATORS ON A SEPARABLE L^p-SPACE[1]

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ABSTRACT. A commutator is a bounded operator which can be expressed as a difference $AB-BA$ using bounded operators A and B. This paper investigates the problem of classifying an operator on a separable L^p-space as either a commutator or a noncommutator. If $1<p<\infty$, we show that compact operators are commutators and that a large class of multiplication operators consists of commutators.

1. Let $\mathfrak{B}$ be a complex Banach space and $\mathfrak{L}(\mathfrak{B})$ the algebra of all (bounded) operators on $\mathfrak{B}$. A *commutator* C in $\mathfrak{L}(\mathfrak{B})$ is an operator such that $C=AB-BA$ for some A, $B\in\mathfrak{L}(\mathfrak{B})$. As shown by Shoda [**6**], an operator on a finite dimensional space is a commutator if and only if the trace is zero. For a Hilbert space $\mathfrak{H}$, a complete description of the set of commutators in $\mathfrak{L}(\mathfrak{H})$ was given by Brown-Pearcy [**3**]. For example, if $\mathfrak{H}$ is infinite dimensional and separable, the only operators which are not commutators are of the form $\lambda I+C$, where λ is a nonzero scalar and C is a compact operator.

This paper describes some commutators in $\mathfrak{L}(\mathfrak{B})$ where $\mathfrak{B}=L^p(X, \mathbf{S}, \mu)$ is a separable infinite dimensional L^p-space for a positive measure space $(X, \mathbf{S}, \mu)$ and $1\leqq p<\infty$. An operator on a Banach space is called *compact* if the closure of the image of the unit ball is a compact set. For $1<p<\infty$, it will be shown that every compact operator on $L^p(X, \mathbf{S}, \mu)$ is a commutator. A *multiplication operator* M_ϕ (or simply *multiplication*) is defined for $\phi\in L^\infty(X, \mathbf{S}, \mu)$ by

$$(M_\phi f)(x) = \phi(x)f(x), \qquad f\in L^p(X).$$

A point λ in the spectrum of an operator is called a *limit point* if either λ is an eigenvalue whose eigenspace is infinite dimensional or λ is a cluster point of the spectrum. A multiplication on l^p or $L^p[0, 1]$, $1<p<\infty$, will be shown to be a commutator if and only if it has two or more limit points in its spectrum or it has zero as its only limit

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point. Similar sufficient conditions are given for multiplications on other L^p-spaces.

The space $L^p[0, 1]$ will be denoted as simply L^p. A result of Bohnenblust [1] is that, for $1 \leqq p < \infty$, each separable infinite dimensional space $L^p(X, \mathbf{S}, \mu)$ is isometrically isomorphic to either (i) l^p, (ii) L^p, (iii) $l^p \oplus L^p$, or (iv) $l^p_n \oplus L^p$ for some natural number n. The isometric isomorphism is determined by the structure of the finite atoms in the measure space (recall E is an *atom* in X if $\mu(F) = \mu(E)$ or $\mu(F) = 0$ for any $F \subset E$). For example, if $(X, \mathbf{S}, \mu)$ has no atoms, then $L^p(X)$ is isometrically isomorphic to L^p. The study of commutators on separable L^p-spaces can thus be restricted to these canonical spaces.

2. **Matricial commutators.** Let $\{L^p(X_i, \mathbf{S}_i, \mu_i)\}$ be a countable family of L^p-spaces for a fixed p, $1 \leqq p < \infty$. The *direct sum* of this family is the set of sequences

$$\sum_{i=1}^{\infty} \oplus L^p(X_i)$$
$$= \left\{(f_1, f_2, \cdots) : f_i \in L^p(X_i),\ i = 1, 2, \cdots, \sum_{i=1}^{\infty} \|f_i\|^p < \infty\right\}.$$

This direct sum is a Banach space with the obvious norm

$$\|(f_1, f_2, \cdots)\| = \left(\sum_{i=1}^{\infty} \|f_i\|^p\right)^{1/p}.$$

Each bounded operator A on the direct sum corresponds to an infinite matrix $(A_{ij})_{i,j=1}^{\infty}$ of bounded linear transformations A_{ij} from $L^p(X_j)$ into $L^p(X_i)$. Addition and multiplication of operators are reflected by the usual matrix operations on the corresponding infinite matrices.

The next two theorems follow matricial constructions given in Brown-Halmos-Pearcy [2] and in Halmos [4].

THEOREM 2.1. *For $1 \leqq p < \infty$, let A be an operator on $\sum_{i=1}^{\infty} \oplus L^p(X, \mathbf{S}, \mu)$, and let $(A_{ij})_{i,j=1}^{\infty}$ be the corresponding matrix of operators. If $\sum_{i,j=1}^{\infty} \|A_{ij}\| < \infty$, then A is a commutator on the direct sum.*

PROOF. Define matrices $(C_{ij})_{i,j=1}^{\infty}$ and $(B_{ij})_{i,j=1}^{\infty}$ by

$$C_{ij} = \delta_{i,j+1} I, \qquad i, j = 1, 2, \cdots,$$

where I is the identity operator on $L^p(X)$, and

$$B_{i1} = 0, \qquad i \geqq 1,$$

$$B_{ij} = \sum_{k=1}^{i} A_{i-k+1,j-k}, \qquad i < j,$$

$$B_{ij} = \sum_{k=1}^{j-1} A_{i-k+1,j-k}, \qquad i \geqq j > 1.$$

An easy computation shows that $(B_{ij})(C_{ij}) - (C_{ij})(B_{ij}) = (A_{ij})$. Further, (C_{ij}) is isometric and therefore defines a bounded operator. It remains to be shown that (B_{ij}) defines a linear transformation on the direct sum which is bounded. Define $\{\alpha_i\}_{i=-\infty}^{\infty}$ by

$$\alpha_i = \sum_{k=1}^{\infty} \| A_{k,k+i-1} \|, \qquad i > 0,$$

$$\alpha_i = \sum_{k=1}^{\infty} \| A_{k-i+1,k} \|, \qquad i \leqq 0.$$

Then let $\beta_{ij} = \alpha_{j-i}$ for $i, j = 1, 2, \cdots$. The matrix $T = (\beta_{ij})_{i,j=1}^{\infty}$ is constant on each diagonal. In addition, $\sum_{i=-\infty}^{\infty} \alpha_i = \sum_{i,j} \|A_{ij}\|$, which is finite by hypothesis. It follows easily that T is a bounded operator on l^1 and l^∞, and by the Riesz convexity theorem, T is also bounded on l^p. Now $\|B_{ij}\| \leqq \beta_{ij}$ for each i and j. Therefore, for $(f_1, f_2, \cdots) \in \sum_{i=1}^{\infty} \oplus L^p(X)$,

$$\|(B_{ij})(f_1, f_2, \cdots)\|^p = \sum_i \left\| \sum_j B_{ij} f_j \right\|^p \leqq \sum_i \left[\sum_j \beta_{ij} \|f_j\| \right]^p$$
$$= \|T(\|f_1\|, \|f_2\|, \cdots)\|^p \leqq \|T\|^p \|(f_1, f_2, \cdots)\|^p.$$

Thus $(B_{ij})(f_1, f_2, \cdots)$ is defined for every vector in the direct sum, and (B_{ij}) represents a bounded operator.

THEOREM 2.2. *Suppose $1 \leqq p < \infty$ and $(X, \mathbf{S}, \mu)$ is a measure space such that $L^p(X)$ is isometrically isomorphic to $\sum_{i=1}^{\infty} \oplus L^p(X)$. Let A, $B \in \mathfrak{L}(L^p(X))$ and define T on $L^p(X) \oplus L^p(X)$ by the matrix $T = \begin{vmatrix} A & 0 \\ B & 0 \end{vmatrix}$. Then T is a commutator.*

PROOF. Since $L^p(X) \oplus [L^p(X)]$ is isometrically isomorphic to

$$L^p(X) \oplus \left[\sum_{i=2}^{\infty} \oplus L^p(X) \right] = \sum_{i=1}^{\infty} \oplus L^p(X),$$

the operator T corresponds to an operator T' on the infinite direct sum whose matrix is

$$\begin{vmatrix} A & 0 & 0 & \cdots \\ B_1 & 0 & 0 & \\ B_2 & 0 & 0 & \\ \vdots & & & \ddots \end{vmatrix},$$

where B_i is an operator on $L^p(X)$ for each i. A calculation shows that this matrix is the commutator of the matrix (C_{ij}) given in the last proof and the matrix

$$\begin{vmatrix} -B_1 & A & 0 & 0 & \cdots \\ -B_2 & 0 & A & 0 & \cdots \\ -B_3 & 0 & 0 & A & \\ \vdots & & & & \ddots \end{vmatrix}.$$

Since A and T' are bounded, this matrix can readily be seen to be bounded. Hence the operators T' and T are commutators.

By a result of Brown-Pearcy [**3**, p. 123], if the sum $SAS^{-1}+TBT^{-1}$ is a commutator, then $A\oplus B$ is a commutator. This fact and the previous theorem make the following proposition useful.

PROPOSITION 2.3. *Let A_1, A_2, A_2, and A_4 be bounded operators on $L^p(X, \mathbf{S}, \mu)$, $1\leqq p<\infty$, such that A_1-A_2 and A_3-A_4 are invertible. Then there exist invertible operators S and T on the direct sum $L^p(X)\oplus L^p(X)$ such that*

$$S\begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix}S^{-1} + T\begin{vmatrix} A_3 & 0 \\ 0 & A_4 \end{vmatrix}T^{-1} = \begin{vmatrix} 0 & C \\ 0 & D \end{vmatrix}$$

for some operators C and D on $L^p(X)$.

PROOF. Let $B=(A_3-A_4)^{-1}$. Then define S and T by

$$S = \begin{vmatrix} I & 0 \\ I & I \end{vmatrix}, \qquad T = \begin{vmatrix} I-(A_1+A_3)B & -(A_1+A_3)B \\ (A_2-A_1)B & (A_2-A_1)B \end{vmatrix}.$$

Computation shows that S and T satisfy the proposition.

The following relationship between canonical separable L^p-spaces with $1<p<\infty$ is often useful.

PROPOSITION 2.4. *If $1<p<\infty$ and n is any natural number, then the spaces L^p, $l^p\oplus L^p$, and $l_n^p\oplus L^p$ are boundedly isomorphic.*

PROOF. Since $1<p<\infty$, the projection P from L^p onto H^p is bounded. Define Φ mapping L^p into $l_1^p\oplus L^p$ by

$$\Phi f = \alpha_0 \oplus [(I - P)f + P(e^{-2\pi i t}f)], \qquad f \in L^p,$$

where α_0 is the 0th Fourier coefficient of f. This mapping is an isomorphism and is bounded. By induction, L^p and $l_n^p \oplus L^p$ are isomorphic. Since L^p is isometrically isomorphic to $\sum_{i=1}^{\infty} \oplus L^p$, one has $l^p \oplus L^p$ isomorphic to $\sum_{i=1}^{\infty} \oplus (l_1^p \oplus L^p)$ and the conclusion follows.

3. **Compact operators as commutators.** Since every infinite dimensional separable space $L^p(X, S, \mu)$ with $1 < p < \infty$ is boundedly isomorphic to l^p or to L^p, it is sufficient to examine compact operators as commutators on the latter two spaces. For simplicity, proofs will be given for only L^p, but the other case is similar.

LEMMA 3.1. *Let C be a compact operator on L^p, $1 < p < \infty$, let $\epsilon > 0$, and let $\{I_n\}$ be a mutually disjoint and exhaustive sequence of measurable subsets of $[0, 1]$. Then there exists a positive integer N such that $\|Cf\| < \epsilon \|f\|$ if $f \in L^p$, $f = 0$ on $\bigcup_{n=1}^{N} I_n$.*

PROOF. Assume the lemma is false. Then for each n, there exists a unit vector f_n in L^p which is zero on $\bigcup_{i=1}^{n} I_i = J_n$ and such that $\|Cf_n\| \geqq \epsilon$. To prove that $\{f_n\}_{n=1}^{\infty}$ converges weakly to zero, let g be in the conjugate space L^q. Then

$$\left| \int_{[0,1]} f_n \bar{g} \, d\mu \right| \leqq \int_{[0,1]-J_n} |f_n| \, |g| \, d\mu \leqq \|f_n\| \left(\int_{[0,1]-J_n} |g|^q d\mu \right)^{1/q};$$

the last term tends to zero by the Lebesgue dominated convergence theorem. Since C is compact and $\{f_n\}$ converges weakly to zero, the sequence $\{Cf_n\}$ converges strongly to zero. However, this contradicts the fact that $\|Cf_n\| \geqq \epsilon$.

LEMMA 3.2. *Suppose $1 < p < \infty$, $(1/p) + (1/q) = 1$, C is a compact operator on L^p, and $\{\epsilon_n\}_{n=1}^{\infty}$ is any sequence of positive numbers. Then there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of mutually disjoint and exhaustive measurable subsets of $[0, 1]$ such that, for each n, $L^p(K_n)$ is isometrically isomorphic to L^p, and*

$$\|Cf\|_p < \epsilon_n \|f\|_p \quad \text{if } f \in L^p, \qquad f = 0 \text{ on } \bigcup_{i=1}^{n} K_i, \tag{3.1}$$

$$\|C^* g\|_q < \epsilon_n \|g\|_q \quad \text{if } g \in L^q, \qquad g = 0 \text{ on } \bigcup_{i=1}^{n} K_i. \tag{3.2}$$

PROOF. Let $\{I_m\}_{m=1}^{\infty}$ be a disjoint, exhaustive sequence of measurable subsets of $[0, 1]$ such that $\mu(I_m) > 0$ for each m. Since C is compact on L^p, the adjoint C^* is compact on L^q. Applying Lemma 3.1

to both C and C^* for each n, one can obtain an increasing sequence of positive integers $\{N_n\}_{n=1}^{\infty}$ satisfying

$$\|Cf\|_p < \epsilon_n\|f\|_p \quad \text{if } f \in L^p \text{ and } f = 0 \text{ on } \bigcup_{m=1}^{N_n} I_m,$$

$$\|C^*g\|_q < \epsilon_n\|g\|_q \quad \text{if } g \in L^q \text{ and } g = 0 \text{ on } \bigcup_{m=1}^{N_n} I_m.$$

So set $K_1 = I_1 \cup I_2 \cup \cdots \cup I_{N_1}$, $K_{n+1} = K_{N_n+1} \cup \cdots \cup K_{N_{n+1}}$ for $n \geqq 1$, and inequalities (3.1) and (3.2) are satisfied. Since K_n has positive measure and is nonatomic in $[0, 1]$, the space $L^p(K_n)$ is isometrically isomorphic to L^p.

THEOREM 3.3. *Let C be a compact operator on a separable infinite dimensional space $L^p(X, S, \mu)$ with $1 < p < \infty$. Then C is a commutator in $\mathfrak{L}(L^p(X))$.*

PROOF. For simplicity, we suppose C is a compact operator on L^p. Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} (2n+1)\epsilon_n < \infty$. Apply Lemma 3.2 to obtain an associated sequence $\{K_n\}_{n=1}^{\infty}$ of subsets of $[0, 1]$. The space L^p is isometrically isomorphic to $\sum_{n=1}^{\infty} \oplus L^p(K_n)$ via the mapping $f \to (f_1, f_2, \cdots)$, where $f_n = f | K_n$ for each n. Under this isomorphism, the operator C on L^p corresponds to an infinite matrix of operators $(C_{ij})_{i,j=1}^{\infty}$ on the direct sum. To characterize C_{ij}, define linear transformations $P_i : L^p \to L^p(K_i)$ and $Q_i : L^p(K_i) \to L^p$ by

$$\begin{aligned} P_i f &= f \,|\, K_i, && && f \in L^p, \\ (Q_i f_i)(x) &= f_i(x), && x \in K_i, && \\ &= 0, && x \notin K_i, && f_i \in L^p(K_i). \end{aligned}$$

Let P_i' and Q_i' be the corresponding projection and injection between L^q and $L^q(K_i)$ (where $1/p + 1/q = 1$). With these definitions, $C_{ij} = P_i C Q_j$ and $C_{ij}^* = P_j' C^* Q_i'$.

We shall show that the entries of $(\|C_{ij}\|)$ are dominated by the entries of the matrix

$$\begin{vmatrix} \|C_{11}\| & \epsilon_1 & \epsilon_2 & \cdots \\ \epsilon_1 & \epsilon_1 & \epsilon_2 & \\ \epsilon_2 & \epsilon_2 & \epsilon_2 & \\ \vdots & & & \ddots \end{vmatrix}.$$

First suppose $j \geqq 2$ and $i \leqq j$. If $f_j \in L^p(K_j)$, then $Q_j f_j = 0$ on $K_1 \cup \cdots \cup K_{j-1}$. By inequality (3.1),

$$\|C_{ij}f_j\|_p = \|P_i C Q_j f_j\| \leqq \|P_i\| \, \|C(Q_j f_j)\| \leqq \epsilon_{j-1}\|Q_j f_j\| \leqq \epsilon_{j-1}\|f_j\|.$$

Hence $\|C_{ij}\| \leqq \epsilon_{j-1}$. Now suppose $i \geqq 2$ and $i > j$. For $g_i \in L^q(K_i)$, $Q_i' g_i = 0$ on $K_1 \cup \cdots \cup K_{i-1}$. Similarly,

$$\|C_{ij}^* g_i\|_q = \|P_j' C^* Q_i' g_i\| \leqq \epsilon_{i-1}\|g_i\|.$$

Therefore $\|C_{ij}\| = \|C_{ij}^*\| \leqq \epsilon_{i-1}$.

Thus the matrix (C_{ij}) satisfies the condition

$$\sum_{i,j=1}^{\infty} \|C_{ij}\| \leqq \|C_{11}\| + \sum_{i=1}^{\infty} (2i+1)\epsilon_i < \infty.$$

Since each space $L^p(K_n)$ is isometrically isomorphic to L^p, Theorem 2.1 is applicable, and the matrix (C_{ij}) is a commutator. Correspondingly the operator C is also a commutator.

4. **Multiplication operators as commutators.** The isometric isomorphism between $L^p(X, \mathbf{S}, \mu)$ and one of the canonical spaces, l^p, L^p, $l^p \oplus L^p$, or $l_n^p \oplus L^p$, may be chosen so that the induced mapping between the algebras of operators transforms multiplications into multiplications. So it is sufficient to investigate multiplications as commutators only on the canonical spaces.

Now operators of the form $\lambda I + C$, where λ is nonzero and C is compact, are not commutators [5]. So we must first characterize these multiplications. Recall that λ in the spectrum of an operator is a limit point if λ is an eigenvalue whose eigenspace is infinite dimensional or λ is a cluster point of the spectrum.

PROPOSITION 4.1. *Let M_ϕ be a multiplication operator on one of the canonical L^p-spaces, $1 \leqq p < \infty$. Then M_ϕ is of the form $\lambda I + C$ with C compact if and only if M_ϕ has exactly the one limit point λ in its spectrum. Hence if $1 < p < \infty$, such a multiplication is a commutator if and only if $\lambda = 0$.*

Now consider multiplications with more than one limit point; again we specialize to the space L^p.

LEMMA 4.2. *Let M_ϕ be a multiplication on L^p, $1 \leqq p < \infty$, and suppose M_ϕ has two or more limit points in its spectrum. Then M_ϕ is a commutator.*

PROOF. Let α and β be distinct limit points in the spectrum of M_ϕ, and set $\epsilon = |\alpha - \beta|/3$. Let U and V be the open balls of radius ϵ about

α and β, respectively. Since α and β are limit points, $\mu(\phi^{-1}U)>0$ and $\mu(\phi^{-1}V)>0$. Therefore, there exist six disjoint and exhaustive subsets $K_1, K_2, \cdots, K_6$ of $[0, 1]$ such that $K_1\cup K_2\subset\phi^{-1}U$, $K_3\cup K_4 \subset\phi^{-1}V$, $\mu(K_i)>0$, $i=1, \cdots, 6$.

Let $\phi_i=\phi|K_i$ for each i. In the space $L^\infty(K_1)$, since $\phi(K_1)\subset U$, $\|\phi_1-\alpha\|_\infty\leqq\epsilon$; similarly, $\|\phi_2-\alpha\|_\infty\leqq\epsilon$, $\|\phi_3-\beta\|_\infty\leqq\epsilon$, and $\|\phi_4-\beta\|_\infty\leqq\epsilon$. Since $\{K_i\}$ is a disjoint and exhaustive family of subsets of $[0, 1]$, L^p is isometrically isomorphic to $\sum_{i=1}^6 \oplus L^p(K_i)$ via the map $f \to(f_1, \cdots, f_6)$, where $f_i=f|K_i$. Under this isomorphism, the operator M_ϕ is transformed to $M_{\phi_1}\oplus \cdots \oplus M_{\phi_6}$ on the direct sum. Since $\mu(K_i)>0$ and K_i is nonatomic, the space $L^p(K_i)$ is isometrically isomorphic to L^p, and the operator M_{ϕ_i} corresponds to a multiplication M_{ψ_i} on L^p for some $\psi_i\in L^\infty[0, 1]$. Thus $\sum\oplus L^p(K_i)$ is isometrically isomorphic to $\sum_{i=1}^6\oplus L^p$, and $M_{\phi_1}\oplus \cdots \oplus M_{\phi_6}$ corresponds to $M_{\psi_1}\oplus \cdots \oplus M_{\psi_6}$.

Now $\|\psi_1-\alpha\|_\infty=\|M_{\psi_1}-M_\alpha\|=\|M_{\phi_1}-M_\alpha\|=\|\phi_1-\alpha\|_\infty\leqq\epsilon$; similarly, $\|\psi_2-\alpha\|_\infty\leqq\epsilon$, $\|\psi_3-\beta\|_\infty\leqq\epsilon$, and $\|\psi_4-\beta\|_\infty\leqq\epsilon$. Therefore, for almost all x,

$$\begin{aligned} |(\psi_1-\psi_3)(x)| &= |\psi_1(x)-\alpha+(\alpha-\beta)+\beta-\psi_3(x)| \\ &\geqq |\alpha-\beta| - \|\psi_1-\alpha\|_\infty - \|\psi_3-\beta\|_\infty \geqq \epsilon, \end{aligned}$$

since $|\alpha-\beta|=3\epsilon$. Also, $|(\psi_2-\psi_4)(x)|\geqq\epsilon$ for almost all x. Consequently the operators $M_{\psi_1}-M_{\psi_3}=M_{\psi_1-\psi_3}$ and $M_{\psi_2}-M_{\psi_4}=M_{\psi_2-\psi_4}$ are both invertible. By Proposition 2.3, there exist invertible operators S and T on $L^p\oplus L^p$ such that

$$S\begin{vmatrix} M_{\psi_1} & 0 \\ 0 & M_{\psi_3}\end{vmatrix}S^{-1}+T\begin{vmatrix} M_{\psi_2} & 0 \\ 0 & M_{\psi_4}\end{vmatrix}T^{-1}=\begin{vmatrix} 0 & A_1 \\ 0 & A_2\end{vmatrix},$$

where A_1 and A_2 are operators on L^p. On the space $(L^p\oplus L^p)\oplus L^p$, let $S_1=S\oplus I$ and $T_1=T\oplus I$. Then one has

$$S_1\begin{vmatrix} M_{\psi_1} & & 0 \\ & M_{\psi_3} & \\ 0 & & M_{\psi_5}\end{vmatrix}S_1^{-1}+T_1\begin{vmatrix} M_{\psi_2} & & 0 \\ & M_{\psi_4} & \\ 0 & & M_{\psi_6}\end{vmatrix}T_1^{-1}=\begin{vmatrix} 0 & A_1 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3\end{vmatrix},$$

for some operator A_3. By Theorem 2.2, the latter matrix represents an operator which is a commutator, because L^p is isometrically isomorphic to $\sum_{i=1}^\infty\oplus L^p$. By Brown-Pearcy [**3**, p. 123], if $SBS^{-1} +TCT^{-1}$ is a commutator, then so is $B\oplus C$. Applying this theorem, one sees that $M_{\psi_1}\oplus M_{\psi_2}\oplus \cdots \oplus M_{\psi_6}$ is a commutator. Under the isomorphisms, the multiplication M_ϕ is also a commutator.

The above proof can easily be modified to show that multiplications on l^p, $1 \leqq p < \infty$, with more than one limit point are commutators, and the use of Proposition 2.4 just before defining S_1 yields the same result for multiplications on $l_n^p \oplus L^p$, $1 < p < \infty$. However, for a multiplication M_ϕ on $l^p \oplus L^p$, $1 < p < \infty$, we assume that $M_\phi | (0 \oplus L^p)$ has two or more limit points (i.e., that M_ϕ is not constant on L^p). This allows K_1, K_2, K_3, and K_4 to be chosen as subsets of $[0, 1]$. Again the use of Proposition 2.4 shows that M_ϕ is a commutator. These remarks yield the following theorem.

THEOREM 4.3. *Let M_ϕ be a multiplication operator on a separable infinite dimensional space $L^p(X, S, \mu)$ with $1 < p < \infty$. If $L^p(X)$ is isometrically isomorphic to l^p, L^p, or $l_n^p \oplus L^p$ for some natural number n, then M_ϕ is a commutator if and only if the spectrum of M_ϕ contains more than one limit point or contains zero as the unique limit point. If $L^p(X)$ is isometrically isomorphic to $l^p \oplus L^p$ via a mapping Φ, then M_ϕ is a commutator if zero is the only limit point in the spectrum of M_ϕ or if $\Phi M_\phi \Phi^{-1} | (0 \oplus L^p)$ has two or more limit points in its spectrum.*

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