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A LIMITATION THEOREM FOR ABSOLUTE SUMMABILITY

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ABSTRACT. Let $A(u)$ be of bounded variation over every finite interval of the nonnegative real axis, and let $\int_0^w e^{-us}\, dA(u)$ be summable $|C, k|$ for a given integer $k \geqq 0$ and a given s whose real part is negative. Then it is known that the function $R(k, w) = (1/\Gamma(k+1)) \cdot \int_w^\infty (u-w)^k\, dA(u)$ (which certainly exists in the $|C, k|$ sense by a well-known summability-factor theorem) satisfies $e^{-ws}w^{-k}R(k, w) = o(1)\ |C, 0|\ (w \to \infty)$. In this paper we extend the above result by showing that if the hypotheses are satisfied with k *fractional*, then $e^{-ws}w^{-k}R(k+\delta, w) = o(1)|\ C, 0|$ for each $\delta > 0$ and that this is best possible in the sense that δ may not be replaced by 0.

1. Let $A(w)$ be of bounded variation over every finite interval of the nonnegative real axis. We write

$$F(a; x) = \int_a^x f(u)dA(u) = L + o(1) \quad (C, k) \tag{1}$$

(read: $F(a; x)$ is summable (C, k) to the limit L, or $\int_a^\infty f(u)\, dA(u)$ exists in the (C, k) sense and equals L) if

$$\Gamma(k + 1)x^{-k}F_k(a; x) = x^{-k}\int_a^x (x - u)^k f(u)dA(u) \to L$$

as $x \to \infty$. (Stieltjes integrals are to be taken in the Riemann sense.) If in addition $x^{-k}F_k(a; x)$ is of bounded variation over $[a, \infty)$ we shall write $|C, k|$ instead of (C, k) in the notations above.

This paper is concerned with the (C, k) and $|C, k|$ summability of

$$C(x) \quad (=C(0; x)) = \int_0^x e^{-us}dA(u) \tag{2}$$

and of

$$R(k', w; x) = 1/\Gamma(k' + 1)\int_w^x (u - w)^{k'}dA(u). \tag{3}$$

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We shall write

$$(4) \qquad R(k', w) = (1/\Gamma(k'+1)) \int_w^\infty (u-w)^{k'} dA(u)$$

so that $R(k', w)$ exists in the (C, k) sense iff (3) is summable (C, k). In virtue of [1, p. 300], if (2) is summable (C, k) (or $|C, k|$) for some $k \geqq 0$, $\operatorname{Re}(s) < 0$, then $R(k', w)$ exists in the (C, k) (or $|C, k|$) sense for each $w \geqq 0$, $k' \geqq 0$. We have now:

THEOREM A [5, pp. 412–413]. *If $k = 0, 1, 2, \cdots$, and (2) is summable $|C, k|$, where $\operatorname{Re}(s) = \sigma < 0$, then*

$$e^{-ws} w^{-k} R(k, w) = o(1) \quad |C, 0|.$$

The last phrase will mean that the function on the left, $g(w)$, say, tends to 0 as $w \to \infty$ and is of bounded variation over $[1, \infty)$, i.e.,

$$\int_1^w dg(u) = -g(1) + o(1) \quad |C, 0|.$$

We state now, writing $[k]$ for the largest integer $\leqq k$, and $\langle k \rangle$ for $k - [k]$:

THEOREM A′. *If k is positive and fractional, and if (2) is summable $|C, k|$ for some s such that $\sigma < 0$, then*

$$e^{-ws} w^{-k} R(k, w) = B(w) + (-1)^{[k]+1} w^{-k} T(w),$$

where $B(w) = o(1)$ $|C, 0|$ and

$$(5) \qquad T(w) = 1/\Gamma(\langle k \rangle) \int_w^{w+1} (u-w)^{\langle k \rangle - 1} C_{[k]}(u) du,$$

$C(u)$ being given by (2).

THEOREM A″. *Under the hypotheses of Theorem A′,*

$$e^{-ws} w^{-k} R(k + \delta, w) = o(1) \quad |C, 0| \qquad \text{for each } \delta > 0.$$

THEOREM A‴. *Under the hypotheses of Theorem A′, $e^{-ws} w^{-k} R(k, w)$ is not necessarily bounded, even with e^{-ws} replaced by e^{-wX} with X as large as we please.*

Theorem A″ is the extension of Theorem A to the case k fractional, and Theorem A‴ shows that Theorem A″ is best possible in the sense that δ may not be replaced by 0.

2. We shall prove the following slight generalization of Theorem A′:

THEOREM A'*. *If $C(w)$ is summable $|C, k-\delta|$, where k is positive and fractional, and $\sigma<0$, $0\leqq\delta<\langle k\rangle$, then*

$$e^{-ws}w^{\delta-k}R(k, w) = B^{(\delta)}(w) + (-1)^{[k]+1}w^{\delta-k}T(w)$$

where $B^{(\delta)}(w)=o(1)$ $|C, 0|$ and $T(w)$ is given by (5).

By [6], slightly modified, the (C) versions of Theorems A'* and A'' (obtained by replacing $|C, \cdots|$ by $(C, \cdots)$) hold. Thus it is sufficient to prove A'* and A'' with '$=o(1)$' replaced by 'is summable'. We shall use (see [6, (25)–(31)]):

LEMMA 1. *If for a given $k\geqq 0$ and $\sigma<0$, $C(w)$ is summable $(C, [k]+1)$, then $R(k, w)$ exists in the $(C, [k]+1)$ sense and*

$$R(k, w) = \sum_{v=0}^{[k]+1} b_v Q(k, v, w)$$

where

$$Q(k, v, w) = \int_w^\infty C_{[k]}(u)(u-w)^{k-v}e^{us}du, \tag{6}$$

the integrals being convergent, and the b's being constants, with

$$b_{[k]+1} = (-1)^{[k]+1}/\Gamma(\langle k\rangle). \tag{7}$$

Theorems A'* and A'' will be deduced from

THEOREM A**. *Under the hypotheses of Theorem A'*, $e^{-ws}w^{\delta-k}\cdot Q(k, v, w)$ is summable $|C, 0|$ if either* (i) $0<\delta<\langle k\rangle$, $v\leqq[k]+1$, *or* (ii) $\delta=0$, $v\leqq[k]$.

We shall require

LEMMA 2. *Let $w\geqq 1$, $-\infty\leqq a<b\leqq\infty$, and $a<u<b$. If*

$$F(w) = \int_a^b g(w, u)f(u)du \quad and \quad \int_1^\infty |d_w g(w, u)| \leqq g(u),$$

then

$$\int_1^\infty |dF(u)| \leqq \int_a^b g(u)|f(u)|du,$$

the integrals over (a, b) being supposed existent in the Lebesgue sense.

PROOF. If $w_0=1<w_1<\cdots<w_m$ we have

$$\sum_{n=1}^m |F(w_n)-F(w_{n-1})| \leqq \int_a^b |f(u)|du \sum_{n=1}^m |g(w_n, u)-g(w_{n-1}, u)|,$$

and the sum on the right is $\leqq g(u)$ by hypothesis.

Proof of Theorem A.** We write

$$(8)\qquad \begin{aligned} p(t) &= t^{\delta-k}C_{k-\delta}(t) \qquad &(t > 0),\\ &= 0 &(t = 0). \end{aligned}$$

Then $p(t)$ is of bounded variation over $[0, \infty)$. Let

$$(9)\qquad D(u, w) = C_{[k]+1}(u) - C_{[k]+1}(w).$$

Then integrating by parts in (6) and using $C_{[k]+1}(u) = O(u^{[k]+1})$, we have $w^{\delta-k}e^{-ws}Q(k, v, w) = (v-k)I_v - sI_{v-1}$ where

$$(10)\qquad I_v = w^{\delta-k}\int_w^\infty (u-w)^{k-v-1}e^{(u-w)s}D(u, w)du.$$

Now $\Gamma(\delta-\langle k\rangle+1)D(u, w)$ can be expressed as

$$(11)\qquad \int_0^u (u-t)^{\delta-\langle k\rangle}C_{k-\delta}(t)dt - \int_0^w (w-t)^{\delta-\langle k\rangle}C_{k-\delta}(t)dt.$$

We write the first integral as the sum of integrals over $[w, u]$ and $[0, w]$ and then combine the second of these with the second integral in (11), thus obtaining $X+Y$, say. We replace $C_{k-\delta}(t)$ by $t^{k-\delta}p(t)$ in each of these, and then put $t = w+(u-w)y$ in X and $t = w-x$ in Y. Inserting the resultant expression in (10) and putting $u = w+z$, we obtain for $\Gamma(\delta-\langle k\rangle+1)I_v$:

$$(12)\qquad \begin{aligned} &\int_0^\infty z^{\delta+[k]-v}e^{zs}dz\int_0^1 (1-y)^{\delta-\langle k\rangle}r(z, y, w)p(w+zy)dy\\ &\quad -\int_0^\infty z^{k-v-1}e^{zs}dz\int_0^w \{x^{\delta-\langle k\rangle} - (x+z)^{\delta-\langle k\rangle}\}(1-x/w)^{k-\delta}p(w-x)dx\\ &\quad = L(w) - M(w), \end{aligned}$$

say, where $r(z, y, w) = (1+zy/w)^{k-\delta}$. Since r decreases as w increases,

$$(13)\qquad \begin{aligned} \int_1^\infty |d_w(rp(w+zy))| \leqq{}& \int_1^\infty |p(\partial r/\partial w)|dw\\ &+ \int_1^\infty r|d_wp(w+zy)| \leqq c(1+z)^{k-\delta}, \end{aligned}$$

say, where c is independent of z and y. Hence by Lemma 2,

$$\int_1^\infty |dL(w)| \leqq c\int_0^\infty (1+z)^{k-\delta}z^{\delta+[k]-v}e^{z\sigma}dz\int_0^1 (1-y)^{\delta-\langle k\rangle}dy,$$

which is finite in either case (i) or case (ii). Next, let

$$q(w, x) = (1 - x/w)^{k-\delta}p(w - x) \qquad (0 \leqq x < w),$$
$$= 0 \qquad (x \geqq w).$$

Then

$$\int_1^\infty \left| d_w q(w, x) \right| \leqq \int_x^\infty \left| d_w((1 - x/w)^{k-\delta}p(w - x)) \right| \leqq c',$$

where c' is independent of x, by an argument similar to (13). Hence by Lemma 2,

$$(14)\quad \int_1^\infty \left| dM(w) \right| \leqq c' \int_0^\infty z^{k-v-1}e^{z\sigma}dz \int_0^\infty \{x^{\delta-\langle k\rangle} - (x + z)^{\delta-\langle k\rangle}\}dx,$$

which is finite in either case (i) or case (ii). Since, finally, each of these cases is satisfied by $v-1$ if it is satisfied by v, the proof of Theorem A** is complete.

Proof of Theorem A″. Put $k-\delta=k'$ in Theorem A**, case (i). Then by Lemma 1 the function $S^{(\delta)}(w)=e^{-ws}w^{-k'}R(k'+\delta, w)$ is of bounded variation over $[1, \infty)$ for each sufficiently small $\delta>0$. Now by [6, Lemma 2] we have, for $p>0$,

$$e^{ws}w^{k'}S^{(\delta+p)}(w) = 1/\Gamma(p) \int_w^\infty (u - w)^{p-1}e^{us}u^{k'}S^{(\delta)}(u)du.$$

The substitution $u=w+x$ followed by an application of (our) Lemma 2 and an argument like that of (13) gives $S^{(\delta+p)}(w)$ of bounded variation over $[1, \infty)$. This completes the proof.

Proof of Theorem A′*. By either case (i) or case (ii) of Theorem A**, together with Lemma 1, we have for $0\leqq\delta<\langle k\rangle$,

$$w^{\delta-k}e^{-ws}R(k, w) = H(w) + w^{\delta-k}e^{-ws}b_{[k]+1}Q(k, [k] + 1, w),$$

where $H(w)$ is of bounded variation over $[1, \infty)$. We write, by (9),

$$Q(k, [k] + 1, w) = \left(\int_w^{w+1} + \int_{w+1}^\infty\right)(u - w)^{\langle k\rangle-1}e^{us}(\partial D/\partial u)du$$
$$= J + K,$$
$$J = e^{ws}\int_w^{w+1} (u - w)^{\langle k\rangle-1}C_{[k]}(u)du$$
$$+ \int_w^{w+1} (u - w)^{\langle k\rangle-1}(e^{us} - e^{ws}) \frac{\partial D}{\partial u} du$$
$$= J_1 + J_2.$$

Integrations by parts of K and J_2, followed by arguments along the lines of (11)–(14), show that $e^{-ws}w^{\delta-k}(K+J_2)$ is of bounded variation over $[1, \infty)$. By (7) this completes the proof.

PROOF OF THEOREM A‴. We shall use

LEMMA 3. *Suppose that k is positive and fractional and that y_n $(n=1, 2, \cdots)$ is a given sequence of positive numbers tending monotonically to ∞. Then there exists a function $C(u)$ such that*

(15)
(a) *$C(u)$ is absolutely continuous over every finite interval of the nonnegative real axis, and $C(0)=0$;*
(b) *$C(u)=o(1)\,|C, k|$;*

but such that the function $T(w)$ given by (5) *satisfies $-T(2n)\geqq c'y_n$ $(n=1,2, \cdots)$ (c' a positive constant).*

PROOF. Let b, c satisfy $0<c-b<2$. We define (compare [3, p. 286]) a function $g_{b,c}(x)$ with domain $b\leqq x\leqq c$, such that it is symmetric about $x=(b+c)/2$ and

$$g_{b,c}(x) = (1 - E^{[k]+2})^{[k]+2} \qquad (b \leqq x \leqq (b + c)/2),$$

where $E=(b+c-2x)/(c-b)$. By induction on r, $g^{(r)}_{b,c}(x)$ has a factor $(1-E^{[k]+2})^{[k]+2-r}E^{[k]+2-r}$ for $b\leqq x\leqq (b+c)/2$ $(r=1, 2, \cdots, [k]+1)$, and thus $g^{(r)}_{b,c}(x)$ is 0 at $x=b$, c, $(b+c)/2$. The latter (with $x=b$, c) is clearly true also for $r=0$. For $r=[k]+2$ the function exists and is bounded in $b<x<(b+c)/2$ and in $(b+c)/2<x<c$. Further,

$$\int_b^{(b+c)/2} | g'_{b,c}(x) | \, dx = \int_{(b+c)/2}^{c} | g'_{b,c}(x) | \, dx = 1. \tag{16}$$

We now write $h_n=\frac{1}{2}e^{-ny_n}$, and define $G(u)$ as follows: for $0\leqq u\leqq 1$, $G(u)=0$; and for $u\geqq 1$ we have, taking $n=1, 2, \cdots$,

$$\begin{aligned} G(u) &= 0 && (2n \leqq u < 2n + 1),\\ &= 1/n && (2n - 1 + h_n \leqq u < 2n - h_n),\\ &= (1/n)g_{2n-1,2n-1+2h_n}(u) && (2n - 1 \leqq u < 2n - 1 + h_n),\\ &= (1/n)g_{2n-2h_n,2n}(u) && (2n - h_n \leqq u < 2n). \end{aligned} \tag{17}$$

Then $0\leqq G(u)\leqq 1$ for all $u>0$. We see that G has a $[k]+1$th derivative everywhere, and a $[k]+2$th derivative almost everywhere, which is bounded on every finite interval. Hence we may choose $C(u)$ such that $C_k(u)=G(u)$, $C(0)=0$, and $C(u)$ is absolutely continuous over every finite interval of the nonnegative real axis. Now by differentiating on the left side we have

$$\int_1^\infty \left|\frac{d}{du}(u^{-k}C_k(u))\right| du \leqq 1 + \sum_{n=1}^\infty \left(\int_{2n-1}^{2n} + \int_{2n}^{2n+1}\right) u^{-k} \mid C_{k'}(u) \mid du.$$

The second integral on the right is 0; and by (17) and (16) the first is $\leqq (2n-1)^{-k}n^{-1}(1+1)$, so that the sum is finite. Hence (15) is established. We now write, by (5),

$$-\Gamma(\langle k\rangle)\Gamma(1-\langle k\rangle)T(2n)$$
$$= -\int_{2n}^{2n+1}(u-2n)^{\langle k\rangle-1}du\int_0^u (u-t)^{-\langle k\rangle}C_{k'}(t)dt.$$

We call this expression I. Replacing u by $2n$ in the inner integral (since $C_{k'}(t)=0$ for $2n\leqq u\leqq 2n+1$), then integrating the latter by parts, and thereafter using the fact that the resulting integral is decreased by replacing its limits by $2n-1+h_n$ and $2n-h_n$, we obtain, after an inversion,

$$\mathrm{I} \geqq \langle k\rangle \int_p^q C_k(t)dt \int_{2n}^{2n+1}(u-2n)^{\langle k\rangle-1}(u-t)^{-\langle k\rangle-1}du,$$

where $p=2n-1+h_n$, $q=2n-h_n$. Writing $u-t$ as

$$\left(1-\frac{2n+1-u}{2n+1-t}\right)(2n+1-t),$$

expanding the $(-\langle k\rangle-1)$th power of the first factor in a binomial series and then integrating term by term, we see that the last inner integral is $\langle k\rangle^{-1}(2n+1-t)^{-\langle k\rangle}(2n-t)^{-1}$. Hence by (17),

$$\mathrm{I} \geqq n^{-1}\int_p^q (2n+1-t)^{-\langle k\rangle}(2n-t)^{-1}dt \geqq n^{-1}2^{-\langle k\rangle}\log\frac{2n-p}{2n-q}.$$

The definitions of p, q, h_n, now give $\mathrm{I}\geqq 2^{-\langle k\rangle}y_n$, which completes the proof.

PROOF OF THEOREM A‴. For the given s, let $y_n = e^{e^{2n}}e^{-2n\sigma}$. Let $C(u)$ satisfy the conditions of Lemma 3, with this y_n.

Choosing $A(u) = \int_0^u e^{ts}dC(t)$, we see that by Theorem A′,

$$R(k, 2n) = e^{2ns}(2n)^k B(2n) + (-1)^{[k]+1}e^{2ns}T(2n),$$

where the term involving $B(2n)$ tends to 0 as $n\to\infty$. But then

$|R(k, 2n)| \geqq c'' e^{e^{2n}}$ for all n large enough, where c'' is a positive constant. This completes the proof.

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