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HIGH CODIMENSIONAL 0-TIGHT MAPS ON SPHERES

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ABSTRACT. For smooth immersions of 2-manifolds into E^M, the condition of 0-tightness is equivalent to that of minimal total absolute curvature, but for higher dimensional manifolds these notions are quite different. By a result of Chern and Lashof, a smooth n-sphere embedded in E^M with minimal total absolute curvature must bound a convex $(n+1)$-cell in an affine $(n+1)$-dimensional subspace, but we show that for any $n>2$ and any $M>n$ there is a 0-tight polyhedral embedding of the n-sphere into E^M with image lying in no hyperplane.

1. **Introduction.** A mapping $f: M\rightarrow E^N$ defined on a connected manifold M is called 0-tight if $\{p\in M \mid (\xi \circ f)(p) \geqq c\} = \xi_c^+(f)$ is connected for every real c and any linear function $\xi: E^N \rightarrow E^1$. For an n-sphere S^n, an immersion $f: S^n \rightarrow E^N$ is said to be *tight* if it has minimal total absolute curvature as defined in the paper of Chern and Lashof [**3**], where it was shown that any such smooth tight immersion must be an embedding of S^n as the boundary of a convex cell with interior in some affine $(n+1)$-space. In the case of a 2-sphere, the condition of 0-tightness implies tightness for smooth immersions and also for simplexwise linear immersions of triangulated 2-spheres. In this note we present a collection of examples to show that for $n>2$, there are 0-tight polyhedral embeddings of S^n which are not tight. The main result is:

THEOREM 3. *For each $n\geqq 3$ and $M>n$ there is a 0-tight embedding of S^n into E^M not lying in any affine hyperplane.*

In [**4**], Kuiper has constructed smooth analogues of these examples.

2. **Preliminaries.** If f is a 0-tight simplexwise linear map defined on a triangulated M, then each of the extreme edges of $\partial Hf(M)$ is the 1-1 image of a 1-dimensional subcomplex in M, where $\partial Hf(M)$ is the convex envelope of $f(M)$ (the boundary of the smallest convex set

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containing $f(M)$), and an extreme edge is a segment which is the intersection of $\partial Hf(M)$ with a support hyperplane (i.e., one which bounds a half-space $\xi_c^+ = \{x \in E^N \mid \xi(x) \geqq c\}$ containing $f(M)$ and such that $\xi_c = \{x \in E^N \mid \xi(x) = c\} \cap f(M) \neq \emptyset$). (Compare [1, Lemma 3.1].) Thus $f^{-1}: (\partial H^1 f(M)) \to M$ gives an embedding into M of the 1-skeleton of the convex envelope of $f(M)$. If all vertices of M have their images in $\partial H^1 f(M)$, then this embedding condition is sufficient as well as necessary in order for f to be 0-tight.

3. **The case** $n = 2$. We first prove an analogue of a theorem of Chern and Lashof for the case where M is the 2-sphere S^2.

THEOREM 1. *If $f: S^2 \to E^N$ is a 0-tight simplexwise linear immersion (i.e., locally 1-1) then f is actually an embedding of S^2 onto the boundary of a convex body with interior in some affine 3-space in E^N.*

PROOF. If σ^2 is a 2-cell in the 2-skeleton of $Hf(S^2)$ then the boundary polygon $\partial\sigma^2$ is an embedded polygon in $f(S^2)$ lying in some hyperplane ξ_c bounding a half-space ξ_c^+ containing $f(S^2)$. If σ^2 were not contained in $f(S^2)$ then each of the complementary components of $\partial\sigma^2$ in $f(S^2)$ would have points, say $f(p_1)$ and $f(p_2)$, in $\xi_c^+ = \{x \in E^N \mid \xi(x) > c\}$ and therefore also in $\xi_{c+\epsilon}^+$ for sufficiently small ϵ. But since any path in $f(S^2)$ from $f(p_1)$ to $f(p_2)$ must pass through $\partial\sigma^2$ which is disjoint from $\xi_{c+\epsilon}^+$, the set $\xi_{c+\epsilon}^+ \cap f(S^2)$ is not connected, contradicting the 0-tightness of f. Therefore σ^2 lies in $f(S^2)$ and $f(S^2)$ contains the entire 2-skeleton of its convex envelope. Moreover $f^{-1}(\partial\sigma^2)$ gives an embedding of the boundary polygon to S^2, so f restricted to one of the complementary components, say C^2, of $f^{-1}(\partial\sigma^2)$ must give an immersion of a disc into the plane which is an embedding on the boundary, so f restricted to C^2 is itself a simplexwise linear embedding onto σ^2. It follows that f^{-1} gives an embedding of the 2-skeleton of $Hf(S^2)$ into f. In particular, the convex set $Hf(S^2)$ must itself be a 3-cell with interior since it is impossible to embed in S^2 the 2-skeleton of the boundary of any convex body with interior of dimension greater than 3.

4. **The case** $n = 3$. We now proceed to show that 0-tightness is not sufficient to give an analogue of this result for higher dimensional spheres. We begin with the following result:

THEOREM 2. *For any $N \geqq 4$ there is a 0-tight simplexwise linear embedding f of the 3-sphere S^3 into E^N so that $f(S^3)$ lies in no affine hyperplane.*

PROOF. We first construct a triangulation S_N^3 of S^3 so that the 1-skeleton of the triangulation is a complete graph with $N+1$ vertices, i.e., every vertex is connected by an edge in the triangulation to every other vertex. We then define f by setting $f(v_i)=a_i$, $i=0, \cdots, N$, where the v_i are the vertices of the triangulation and the a_i are the vertices of a regular N-simplex Δ^N in E^N with center situated at the origin of E^N. We then extend f linearly over the simplexes of S_N^3 of dimensions 1, 2, and 3 to obtain an embedding with all vertices of S_N^3 going to extreme vertices and with the 1-skeleton of the convex envelope $\partial Hf(S^3)=\Delta^N$ being contained in $f(S^3)$, so we have a 0-tight embedding not contained in any affine hyperplane. The existence of a triangulation S_N^3 of S^3 whose 1-skeleton is a complete graph with $N+1$ vertices follows from a lemma:

LEMMA 1. *For any $N\geqq 4$ there is a combinatorial 3-ball B_N^3 with N vertices, all contained on the boundary 2-sphere ∂B_N^3 such that the 1-skeleton of B_N^3 is a complete graph with N vertices and such that the polygon $[v_0, v_1, \cdots, v_{N-1}]$ is an embedded polygon on ∂B_N^3.*

PROOF OF LEMMA 1. For $N=4$, we take the 3-simplex Δ^3 itself to be B_4^3, and since the quadrilateral $[v_0, v_1, v_2, v_3]$ is an embedded polygon on the boundary, the conditions of the lemma are satisfied.

If we assume the result for N, then we may let C_N^2 denote one of the complementary triangulated 2-cells in B_N^3 bounded by the polygon $[v_0, v_1, \cdots, v_{N-1}]$. We then set B_{N+1}^3 to be the union of B_N^3 and the cone $v_N C_N^2$ over C_N^2 from a new vertex v_N, the two 3-cells being identified along the 2-cell C_N^2 common to their boundaries. The 1-skeleton of B_{N+1}^3 then consists of that of B_N^3 together with all edges $v_i v_N$, $i=0, \cdots, N-1$, so it is the complete graph on $(N+1)$ vertices, and the polygon $[v_0, v_1, \cdots, v_{N-1}, v_N]$ is embedded so the lemma is complete.

Using the lemma we may obtain a triangulation S_N^3 of S^3 whose 1-skeleton is the complete graph on $N+1$ vertices by attaching B_N^3 to the 3-cell $v_N(B_N^3)$, obtained by taking the cone over the 2-sphere B_N^3, along this common boundary 2-sphere. The proof of the theorem is thus complete.

5. **The case $n>3$.** In order to obtain 0-tight maps of higher dimensional spheres S^{k+3} into Euclidean spaces E^{k+N}, we suspend the maps constructed in the previous theorem in such a way that the resulting maps remain 0-tight. Consider N to be fixed in this argument and let e_j and $-e_j$ denote unit vectors orthogonal to E^{j-1+N} in

E^{j+N} for $j=1, 2, \cdots, k$. The suspension $S(S_N^3)$ is defined to be the 4-sphere obtained by identifying the two 4-cells given by the cones $u_1S_N^3$ and $w_1S_N^3$ over their common boundary 3-sphere S_N^3. We then define the jth suspension by $S^j(S_N^3)=S(S^{j-1}(S_N^3))$ for $j>1$ and we observe that this jth suspension gives a triangulation of the $j+3$ sphere, which we denote by S_N^{j+3}. We then define a simplexwise linear map $g: S_N^{j+3}\to E^{j+N}$ by sending S_N^3 to $E^N\subset E^{j+N}$ by the map f constructed above and by setting $g(u_j)=e_j$, $g(w_j)=-e_j$ and extending linearly over all simplexes of S_N^{j+3}. The images of these maps map the $j+3$ sphere into the convex $(j+N)$-cell $S^j(\Delta^N)$ given by the j-fold geometric suspension of the N-simplex in E^{j+N}, and the image of $g(S_N^{j+3})$ contains all of the edges in the 1-skeleton of this $(j+N)$-cell, so we have proved the following:

THEOREM 3. *For each $n\geqq 3$ and $M>n$, there is a* 0-*tight embedding of S^n into E^M, not lying in any affine hyperplane.*

6. **Remarks.** We remark now that none of these mappings $f: S^3\to E^N$ for $N>4$ is 1-tight, where a map $f: M\to E^N$ is said to be k-*tight* if any k-dimensional cycle N^{k-1} in $\xi_c^+(f)$ which bounds in M also bounds in $\xi_c^+(f)$ for all real c and all linear functions $\xi: E^N\to E^1$. (By "bounds" in the previous sentence we mean "is the boundary of some chain P^k of dimension k in M" in the appropriate homology theory, for example singular homology theory, or a theory which requires that P^k be an embedded k-manifold-with-boundary if N^{k-1} is also embedded. Cf. [**2**] for a discussion of this notion.)

In particular, if $f: S^3\to E^N$ is 1-tight, then since every 1-cycle in S^3 bounds, it follows that every 2-cell in the convex envelope $Hf(S^3)$ must be contained in $f(S^3)$, and by an argument similar to that used in the first theorem, f^{-1} gives an embedding of the 2-skeleton of $\partial Hf(S^3)$ into S^3. For the maps constructed above, this would mean that we would have an embedding of the 2-skeleton of Δ^N into S^3, but for $N\geqq 5$, such an embedding is impossible. (A proof of the impossibility of such an embedding follows by the same argument used to show that it is impossible to embed the 1-skeleton of the 4-simplex in E^2, and in general it follows that it is impossible to embed the j-skeleton of the $(j+3)$-simplex in E^{j+1}.) More generally, since the complex $K_{j,N}$ consisting of the j-skeleton of the N-simplex cannot be embedded in E^{N-1} if $2j+1\geqq N\geqq j+2$ (cf. [**5**, p. 114]) it follows that if $j\geqq(k+1)/2$, there is no j-tight embedding of S^k into E^{k+2} (not lying in a hyperplane) so that $f(S^k)$ is a subcomplex of Δ^{k+2} in E^{k+2}.

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