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ON MODULI OF CONTINUITY AND DIVERGENCE OF FOURIER SERIES ON GROUPS

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ABSTRACT. Let G be a 0-dimensional, metrizable, compact, abelian group. Then its character group X is a countable, discrete, torsion, abelian group. N. Ja. Vilenkin defined an enumeration for the elements of X and developed part of the Fourier theory on G. Among other things he proved on G a theorem similar to the Dini-Lipschitz test for trigonometric Fourier series. In this note we will show that Vilenkin's result is in some sense the best possible by proving the existence of a continuous function f on G whose modulus of continuity, $\theta_k(f)$, satisfies $\theta_k(f)=O(k^{-1})$ as $k\rightarrow\infty$ and whose Fourier series diverges at a point of G. The function f will be defined by means of the analogue in X of the classical Fejér polynomials.

Throughout this paper we will use the terminology and notations of [2] or [4] and we will assume that the reader is familiar with at least one of these papers.

Let G and X be as in the abstract. The best-known example of such a group G is $\prod_{n=1}^{\infty}(Z(2))_n$, which has the system of Walsh functions as its dual group (see [1]). Vilenkin's theorem, mentioned in the abstract, is the following [4, 3.5]: If G is a primary group and if f is a continuous function on G whose modulus of continuity, $\theta_k(f)$, satisfies $\theta_k(f)=o(k^{-1})$ as $k\rightarrow\infty$, then the Fourier series of f converges uniformly. In [2, Corollary 2] Onneweer and Waterman obtained the same result for groups G which satisfy the condition that $\sup_n p_n=p<\infty$. From now on we will only consider groups G for which $\sup_n p_n=p<\infty$. We establish the following theorem.

THEOREM. *There exists a continuous function f on G such that* (i) *the Fourier series of f does not converge at $0\in G$ and* (ii) $\theta_k(f)=O(k^{-1})$ *as $k\rightarrow\infty$.*

The proof resembles G. Faber's proof of a similar theorem for trigonometric Fourier series [5, p. 302]. It will be preceded by two lemmas.

LEMMA 1. *Let $a_0=1$ and if n satisfies $m_k\leqq n<m_{k+1}$ for some k, let $a_n=(-1)^k(m_{k+1}-m_k)^{-1}$. Let $S(x)=\sum_{n=0}^{\infty}a_n\chi_n(x)$.*

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Then the partial sums of $S(x)$, $S_m(x;S)=\sum_{n=0}^{m-1} a_n\chi_n(x)$, *satisfy*

$$|S_m(x;S)| \leqq 4$$

uniformly in m *and in* $x\in G$.

PROOF. Let m be any natural number and take s so that $m_s<m\leqq m_{s+1}$. Set $m=b_sm_s+m'$, with $0<b_s\leqq p_{s+1}$ and $0\leqq m'<m_s$. Then

$$\begin{aligned} S_m(x;S) &= \sum_{i=0}^{m-1} a_i\chi_i(x) \\ &= 1+\sum_{k=0}^{s-1}\frac{(-1)^k}{m_{k+1}-m_k}\sum_{i=m_k}^{m_{k+1}-1}\chi_i(x)+\sum_{i=m_s}^{m-1}\frac{(-1)^s}{m_{s+1}-m_s}\chi_i(x) \\ &= 1+B_1+B_2. \end{aligned}$$

Next we observe that, for each k,

$$\begin{aligned} \sum_{i=m_k}^{m_{k+1}-1}\chi_i(x) &= \sum_{j=1}^{p_{k+1}-1}\sum_{i=jm_k}^{(j+1)m_k-1}\chi_i(x) = \sum_{j=1}^{p_{k+1}-1}\sum_{i=0}^{m_k-1}\chi_{jm_k+i}(x) \\ &= \sum_{j=1}^{p_{k+1}-1}\chi_{m_k}^{j}(x)\sum_{i=0}^{m_k-1}\chi_i(x) = \sum_{j=1}^{p_{k+1}-1}\chi_{m_k}^{j}(x)D_{m_k}(x). \end{aligned}$$

Also, according to [4, 2.2] we have, for each r,

(i) if $x\in G_r$, then $D_{m_r}(x)=m_r$ and $\chi_{m_r}(x)=\exp(2\pi i\alpha(x)/p_{r+1})$, where $\alpha(x)\in\{0,1,\cdots,p_{r+1}-1\}$ and $\alpha(x)=0$ iff $x\in G_{r+1}$;

(ii) if $x\notin G_r$, then $D_{m_r}(x)=0$;

(iii) if $x\in G_r$, then $\chi_{m_i}(x)=1$ for $0\leqq i<r$.

Since $x\in G$ implies $x\in G_r\backslash G_{r+1}$ for some r, we have

$$\begin{aligned} |B_1| &= \left|\sum_{k=0}^{s-1}\frac{(-1)^k}{m_{k+1}-m_k}\sum_{j=1}^{p_{k+1}-1}\chi_{m_k}^{j}(x)D_{m_k}(x)\right| \\ &= \left|\sum_{k=0}^{r}\frac{(-1)^k}{m_{k+1}-m_k}m_k\sum_{j=1}^{p_{k+1}-1}\chi_{m_k}^{j}(x)\right| \\ &\leqq \left|\sum_{k=0}^{r-1}\frac{(-1)^k}{p_{k+1}-1}(p_{k+1}-1)\right|+\left|\frac{(-1)^r}{p_{r+1}-1}(-1)\right| \\ &= \left|\sum_{k=0}^{r-1}(-1)^k\right|+\left|\frac{1}{p_{r+1}-1}\right|\leqq 2. \end{aligned}$$

For B_2 we have

$$|B_2| \leqq \frac{1}{m_{s+1}-m_s}\sum_{i=m_s}^{m-1}|\chi_i(x)| \leqq \frac{m-m_s}{m_{s+1}-m_s}\leqq 1.$$

Consequently, $|S_m(x; S)| \leqq 4$, uniformly in $x \in G$ and for all m.

In the next lemma we define the Fejér polynomials $Q_n(x)$ in X. The definition is the same as that given by F. Schipp [3] for the case of the Walsh functions.

LEMMA 2. *For each n, let $\mu_n = \sum_{k=0}^{n-1} z_k m_k$, with $z_k = 0$ if k is odd and $z_k = 1$ if k is even. Let*

$$Q_n(x) = \chi_{\mu_n}(x) \sum_{k=0}^{m_n - 1} a_k \chi_k(x).$$

Then

(i) $S_{\mu_n}(0; Q_n) > (n-2)/2p$ *and*

(ii) $|Q_n(x)| \leqq 4$ $(x \in G)$.

PROOF. For any two nonnegative integers k and l, let $k \oplus l$ be defined by the property that $\chi_k(x)\chi_l(x) = \chi_{k\oplus l}(x)$. Then, clearly, the Fourier series of $Q_n(x)$ is the series

$$\sum_{k=0}^{m_n - 1} a_k \chi_{k \oplus \mu_n}(x).$$

Also, since for each s, $\chi_{m_s}^{p_s+1} = \chi_t$ for some $t < m_s$ [4, 2.2], it follows that if k is of the form $k = (p_{2s+1} - 1)m_{2s} + k'$ with $0 \leqq k' < m_{2s}$ and some $s \in \{0, 1, \cdots, [(n-1)/2]\}$, then $a_k = (m_{2s+1} - m_{2s})^{-1}$ and $k \oplus \mu_n < \mu_n$. Moreover, these are the only values of k, $0 \leqq k < m_n$, for which $k \oplus \mu_n < \mu_n$. Consequently,

$$S_{\mu_n}(0; Q_n) = \sum_{s=0}^{[(n-1)/2]} \frac{1}{m_{2s+1} - m_{2s}} m_{2s} = \sum_{s=0}^{[(n-1)/2]} \frac{1}{p_{2s+1} - 1}$$
$$> \frac{1}{p}\left[\frac{n-1}{2}\right] \geqq \frac{n-2}{2p}.$$

The second inequality of Lemma 2 follows immediately from Lemma 1.

PROOF OF THE THEOREM. Let the function f be defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k'} \chi_{m_{k'}}(x) Q_{k'}(x),$$

where we write k' for 2^k. Since for each k the Fourier series of $\chi_{m_{k'}}(x)Q_{k'}(x)$ contains only characters χ_s with $m_{k'} \leqq s < 2m_{k'} < m_{(k+1)'}$, different values of k correspond to different sets of characters in X. Consequently, Lemma 2(i) implies that, for each $k > 1$,

$$\left| S_{m_{k'}+\mu_{k'}}(0;f) - S_{m_{k'}}(0;f) \right| = \left| \frac{1}{k'} S_{\mu_{k'}}(0;\, Q_{k'}) \right|$$

$$> \frac{1}{k'} \cdot \frac{k'-2}{2p} = \frac{1}{2p} - \frac{1}{pk'} \geqq \frac{1}{4p}\,.$$

Therefore, the Fourier series of f does not converge at $0 \in G$.

Next we prove part (ii) of the theorem. Take any natural number l and let $y \in G_l$. Choose $\bar{s}$ so that, for all $k \leqq \bar{s}$, the Fourier series of $\chi_{m_{k'}}(x)Q_{k'}(x)$ contains only characters $\chi_t(x)$ with $t < m_l$, whereas the Fourier series of $\chi_{m_{(\bar{s}+1)'}}(x)Q_{(\bar{s}+1)'}(x)$ contains at least one $\chi_t(x)$ with $t \geqq m_l$. Then $m_{(\bar{s}+2)'} \geqq m_l$, i.e., $2^{-(\bar{s}+2)} \leqq l^{-1}$. Also, since $y \in G_l$, $\chi_t(x+y) = \chi_t(x)$ for all $x \in G$ and $0 \leqq t < m_l$. Therefore,

$$\begin{aligned} |f(x+y) - f(x)| \leqq & \left| \sum_{k=1}^{\bar{s}} \frac{1}{k'} \left[\chi_{m_{k'}}(x+y)Q_{k'}(x+y) - \chi_{m_{k'}}(x)Q_{k'}(x) \right] \right| \\ & + \sum_{k=\bar{s}+1}^{\infty} \frac{1}{k'} \left| \chi_{m_{k'}}(x+y) \right| \left| Q_{k'}(x+y) \right| \\ & + \sum_{k=\bar{s}+1}^{\infty} \frac{1}{k'} \left| \chi_{m_{k'}}(x) \right| \left| Q_{k'}(x) \right| \\ \leqq & \; 0 + \sum_{k=\bar{s}+1}^{\infty} \frac{1}{k'} \left| Q_{k'}(x+y) \right| + \sum_{k=\bar{s}+1}^{\infty} \frac{1}{k'} \left| Q_{k'}(x) \right|. \end{aligned}$$

Lemma 2(ii) implies that

$$|f(x+y) - f(x)| \leqq 2 \cdot 4 \sum_{k=\bar{s}+1}^{\infty} \frac{1}{2^k} = 8\, \frac{1}{2^{\bar{s}}} = O\left(\frac{1}{l}\right).$$

Hence $\theta_l(f) = O(l^{-1})$. This completes the proof of the theorem.

References

1. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414. MR **11**, 352.

2. C. W. Onneweer and Daniel Waterman, *On uniform convergence of Fourier series on groups*. I, Michigan Math. J. (to appear).

3. F. Schipp, *Bemerkung zur Divergenz der Walsh-Fourierreihen*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **11** (1968), 53–58. MR **39** #3226.

4. N. Ja. Vilenkin, *On a class of complete orthonormal systems*, Izv. Akad. Nauk SSSR Ser. Mat. **11** (1947), 363–400; English transl., Amer. Math. Soc. Transl. (2) **28** (1963), 1–35. MR **9**, 224; MR **27** #4001.

5. A. Zygmund, *Trigonometric series*. Vol. 1, 2nd ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

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