PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, No. 2, July 1971

IDENTITIES OF GROUP ALGEBRAS

PAUL MILNES[1]

ABSTRACT. In this note the converse to the following well-known proposition about locally compact topological groups G is proved: if G is discrete, then the group C^*-algebra $C^*(G)$ has an identity.

Let G be a locally compact topological group and let $L:f\to L_f$ denote the left regular representation of $L^1(G)$ on $L^2(G)$; $L_fh=f^*h, \forall h\in L^2(G)$. If G is compact, it is a classical result that L_f is a compact operator $\forall f\in L^1(G)$. If, as well, G is not discrete, then $L^2(G)$ is infinite dimensional and, for each $f\in L^1(G)$, there is a sequence in $\{L_fh/h\in L^2(G),\ \|h\|_2=1\}$ converging to zero in $L^2(G)$. We now prove this last statement for arbitrary nondiscrete locally compact topological groups.

PROPOSITION. *Suppose G is a nondiscrete locally compact topological group. Then, if $f\in L^1(G)$, there is a sequence in $\{L_fh/h\in L^2(G), \|h\|_2=1\}$ converging to zero in $L^2(G)$.*

PROOF. It suffices to prove the result for $f\in C_{00}(G)$, a norm-dense subset of $L^1(G)$. Suppose $f\in C_{00}(G)$, $\|f\|_\infty=a$ and the support of f is K, a compactum. Let $\{V_n\}$ be a decreasing sequence of compact symmetric neighbourhoods of the identity of G such that $\mu(V_n)\to 0$, where μ is left Haar measure on G, and let $h_n=\chi_{V_n}/(\mu(V_n))^{1/2}$. Then $L_fh_n(t)=f^*h_n(t)=\int f(s)h_n(s^{-1}t)d\mu(s)=0\ \ \forall t\notin KV_n$ and, $\forall t\in G$, $|L_fh_n(t)|\leq a(\mu(V_n))^{1/2}$. Hence, $\|L_fh_n\|_2^2\leq a^2\mu(V_n)\mu(KV_n)\to 0$ as $n\to\infty$. Q.E.D.

Let $C_r^*(G)$ denote the C^*-algebra obtained by completing $L^1(G)$ in the norm $f\to\|L_f\|$. $C_r^*(G)$ can be identified with the C^*-subalgebra of $\mathcal{L}(L^2(G))$ generated by the operators $\{L_f/f\in L^1(G)\}$ (see [1, p. 187], for example).

COROLLARY 1. *Suppose G is a nondiscrete locally compact topological group. Then $C_r^*(G)$ does not have an identity.*

PROOF. Suppose $C_r^*(G)$ does have an identity, 1. Let $f\in L^1(G)$ be such that $\|1-L_f\|\leq 1/4$ and let $h\in L^2(G)$, $\|h\|_2=1$ be such

Received by the editors October 26, 1970.

AMS 1970 subject classifications. Primary 22D25.

Key words and phrases. Locally compact topological group, C^*-algebra.

[1] The author is indebted to Professor L. T. Gardner, his supervisor at the University of Toronto, where this research was done, for some helpful discussions.

that $\|L_f h\|_2 \leqq 1/4$. Then $\|(1-L_f)h\|_2 = \|h - L_f h\|_2 \geqq \|h\|_2 - \|L_f h\|_2 \geqq 3/4$, which is a contradiction.

The group C^*-algebra $C^*(G)$ of G is obtained by completing $L^1(G)$ in the norm $f \to \sup\|\pi_f\|$, where π ranges over all $*$-representations of $L^1(G)$ as operators on a Hilbert space.

COROLLARY 2. *Suppose G is a nondiscrete locally compact topological group. Then $C^*(G)$ does not have an identity.*

PROOF. Since $C_r^*(G)$ is a homomorphic image of $C^*(G)$ [1, (1.15), p. 187] and $C_r^*(G)$ has no identity, $C^*(G)$ has no identity.

It is well known that, if G is discrete, $L^1(G)$ has an identity and hence so do $C_r^*(G)$ and $C^*(G)$.

REFERENCES

1. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236. MR **37** #4208.

UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA