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IRREDUCIBLE LIE ALGEBRAS OF INFINITE TYPE

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ABSTRACT. Let V be a finite dimensional vector space over an algebraically closed field of characteristic $\neq 2, 3, 5$. It is shown that if $L \subseteq \mathrm{gl}(V)$ is an irreducible Lie algebra of infinite type then either $L = \mathrm{gl}(V)$, $L = \mathrm{sl}(V)$, $\dim V = 2r \geqq 4$ and $L = \mathrm{sp}(V)$, $\dim V = 2r \geqq 4$ and $L = \mathrm{csp}(V)$, or there exists $A \in L$ such that $\mathrm{ad}\, A \neq 0 = (\mathrm{ad}\, A)^2$. As a corollary we obtain E. Cartan's classification of the irreducible Lie algebras of infinite type over $\mathbf{C}$.

Let L be a Lie algebra of linear transformations of a vector space V. For each nonnegative integer n the nth *Cartan prolongation*, L_n, is defined inductively by $L_0 = L$ and

$$L_n = \{\phi \in \mathrm{Hom}(V, L_{n-1}) \mid y(x\phi) = x(y\phi) \text{ for all } x, y \in V\}$$

for $n \geqq 1$. If $L_n \neq (0)$ for all $n \geqq 0$ then L is said to be of *infinite type*. The main result of this paper is:

THEOREM 1. *Let V be a finite dimensional vector space over an algebraically closed field of characteristic $\neq 2,3,5$. If L is an irreducible Lie algebra of infinite type then either $L = \mathrm{gl}(V)$, $L = \mathrm{sl}(V)$, $\dim V = 2r \geqq 4$ and $L = \mathrm{sp}(V)$, $\dim V = 2r \geqq 4$ and $L = \mathrm{csp}(V)$, or there exists an $A \in L$ such that* $\mathrm{ad}\, A \neq 0 = (\mathrm{ad}\, A)^2$.

Now it is easily seen (as in [7]) that if $(\mathrm{ad}\, A)^2 = 0$ then A belongs to the radical of the Killing form. If $\Phi = \mathbf{C}$ and L is reductive this implies that $\mathrm{ad}\, A = 0$. Furthermore, it is known (Theorem 1 of [4]) that an irreducible Lie algebra over $\mathbf{C}$ is of infinite type if and only if $L_2 \neq (0)$. Thus Theorem 1 implies the following theorem of E. Cartan [1]:

THEOREM 2. *Let V be a finite dimensional vector space over $\mathbf{C}$ and let L be an irreducible Lie algebra of linear transformations of V such that $L_2 \neq (0)$. Then either $L = \mathrm{gl}(V)$, $L = \mathrm{sl}(V)$, $\dim V = 2r \geqq 4$ and $L = \mathrm{sp}(V)$, or $\dim V = 2r \geqq 4$ and $L = \mathrm{csp}(V)$.*

Theorem 2, which is important in the study of primitive pseudogroups and infinite Lie algebras of Cartan type, has been proved by several authors ([1], [2], [3], [5], [9]). These proofs have involved considerable use of the classification and representation theory of

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semisimple Lie algebras over $\mathbf{C}$ and hence cannot be generalized to fields of prime characteristic. In the proof presented here we use more elementary techniques which are valid over algebraically closed fields of characteristic $\neq 2, 3, 5$.

We will consider the following three conditions on a Lie algebra L of linear transformations of a vector space V:

CONDITION A. There exists $A \in L$ with rank $A = 1$ and $A^2 \neq 0$.

CONDITION B. Either $\dim V = 2$ or there exist $A, B \in L$ with rank A $=$ rank $B = 1$, ker $A =$ ker B, and $VA \neq VB$.

CONDITION C. There exists $A \in L$ with ad $A \neq 0 = (\text{ad } A)^2$.

By Condition $\sim$X we will mean the negation of Condition X.

Theorem 1 is clearly a consequence of the following two lemmas.

LEMMA 1. *If Φ is an algebraically closed field of arbitrary characteristic, V is a finite dimensional vector space over Φ, and $L \subseteq \mathrm{gl}(V)$ is a Lie algebra of infinite type then L contains a rank one transformation.*

LEMMA 2. *Let V be a finite dimensional vector space over a field Φ of characteristic $\neq 2, 3, 5$. Let L be an irreducible Lie algebra of linear transformations of V. Assume that L contains a rank one transformation. Then:*

(i) *If Condition* A *holds* $L = \mathrm{gl}(V)$.

(ii) *If Conditions* $\sim$A, B, *and* $\sim$C *hold then* $L = \mathrm{sl}(V)$.

(iii) *If Conditions* $\sim$A, $\sim$B, *and* $\sim$C *hold then* $\dim V = 2r \geqq 4$ *and* $L = \mathrm{sp}(V)$ *or* $L = \mathrm{csp}(V)$.

It is shown in [2] that if $\Phi = \mathbf{C}$ Lemma 1 is a consequence of Hilbert's Nullstellensatz. The proof given there is in fact independent of the assumption $\Phi = \mathbf{C}$ and could be used to prove our Lemma 1. We will give a somewhat more elementary proof here. Part (i) of Lemma 2 is proved in [6] for the case $\Phi = \mathbf{C}$.

In the proof of Lemma 1 we will need:

LEMMA 3. *Let Φ be an algebraically closed field of arbitrary characteristic. Let V and W be finite dimensional vector spaces over Φ with $2 \leqq \dim V \leqq \dim W$. Let T be a subspace of* $\mathrm{Hom}(V, W)$ *such that* $\dim T \geqq \dim W$. *Then there exist $\phi \in T$ and $v \in V$ such that $\phi \neq 0$, $v \neq 0$, and $v\phi = 0$.*

PROOF. Suppose $T \subseteq \mathrm{Hom}(V, W)$ is a counterexample to the lemma such that $\dim W$ is minimal among all counterexamples. As T is a counterexample we have, for any nonzero $v \in V$, $\dim W \geqq \dim vT$ $= \dim T \geqq \dim W$. Thus for any $w \in W$ there is a unique $\phi \in T$ such that $v\phi = w$. Then if v_1, v_2 are linearly independent elements of V we can find a basis $\{\psi_1, \cdots, \psi_n\}$ of T such that

$$(1)\qquad \begin{aligned} v_2\psi_i &= v_1\psi_{i+1} \quad \text{for all } 1 \leqq i \leqq n-1, \quad \text{and} \\ v_2\psi_n &= \sum_{i=1}^{n} v_1\psi_i a_i \quad \text{where the } a_i \in \Phi. \end{aligned}$$

(For if linearly independent elements $\psi_1, \cdots, \psi_j$ satisfying (1) have been found for some $1 \leqq j \leqq n-1$ then the minimality of $\dim W$ implies that $v_2\psi_j \notin \langle v_1\psi_1, \cdots, v_1\psi_j\rangle$. Thus there exists a unique $\psi_{j+1} \in T$ such that $\psi_1, \cdots, \psi_{j+1}$ are linearly independent and satisfy (1). Proceeding by induction on j gives the result.) Now there exists $\lambda \in \Phi$ such that $\lambda^n - \sum_{i=0}^{n-1} \lambda^i a_{i+1} = 0$. Setting $b_k = \lambda^{n-k} - \sum_{i=0}^{n-k-1} \lambda^i a_{k+i+1}$ for $1 \leqq k \leqq n-1$ we see that $\psi = \psi_n + \sum_{i=1}^{n-1} \psi_i b_i \in T$ and $(v_2 - \lambda v_1)\psi = 0$. This contradicts the choice of T and proves the lemma.

Proof of Lemma 1. For $\phi \in L_i$ and $j \leqq i$ define

$$\mathrm{im}_j(\phi) = \langle v_j(\cdots(v_1\phi)\cdots) \mid v_1, \cdots, v_j \in V\rangle.$$

Now if $x \in \ker \phi$ and $y \in V$ then $0 = y(x\phi) = x(y\phi)$ so we have $\ker \phi \subseteq \ker(y\phi)$. Thus if $\psi \in \mathrm{im}_j(\phi)$ we have $\ker \psi \supseteq \ker \phi$. Furthermore if $d_i = \min\{\text{rank } \phi \mid 0 \neq \phi \in L_i\}$ we have $d_0 \leqq d_1 \leqq \cdots \leqq \dim V$. Hence there is some integer N such that $d_N = d_{N+i}$ for all $i \geqq 0$. Now if $i \geqq j \geqq 0$ and if $\phi \in L_{N+i}$ satisfies rank $\phi = d_N$ then for $0 \neq \psi \in \mathrm{im}_j(\phi)$ we have $\ker \psi \supseteq \ker \phi$ and rank $\psi \geqq d_N = \text{rank } \phi$. Thus $\ker \psi = \ker \phi$. Thus $\mathrm{im}_j(\phi) \subseteq \mathrm{Hom}(V/\ker \phi, \mathrm{im}_{j+1}(\phi))$ and $0 \neq \psi \in \mathrm{im}_j(\phi)$ implies rank $\psi = d_N = \dim(V/\ker \phi)$. Thus if $d_N \geqq 2$, Lemma 3 shows that $\dim \mathrm{im}_j(\phi) \leqq \dim \mathrm{im}_{j+1}(\phi) - 1$. Hence $\dim L_{N-1} \geqq \dim \mathrm{im}_{i+1}(\phi) \geqq \dim \mathrm{im}_1(\phi) + i \geqq i$ for all $i \geqq 0$. Since L_{N-1} is finite dimensional this is impossible. Hence $1 = d_N = d_0$, proving the lemma.

Proof of Lemma 2. Let $n = \dim V$. If $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ are bases for V we define elements E_{ij} and $F_{ij} \in \mathrm{gl}(V)$ for $1 \leqq i, j \leqq n$ by $x_k E_{ij} = \delta_{ki} x_j$ and $y_k F_{ij} = \delta_{ki} y_j$. If $2k \leqq n$ we define $\mathrm{sp}(x_1, \cdots, x_{2k})$ to be the Lie algebra of all $A \in \mathrm{gl}(V)$ such that $VA \subseteq \langle x_1, \cdots, x_{2k}\rangle$, $x_r A = 0$ for all $r > 2k$, and A is skew with respect to the skew-symmetric bilinear form defined by $(x_{2i+1}, x_j) = \delta_{j,2i+2}$, $(x_{2i+2}, x_j) = -\delta_{j,2i+1}$, and $(x_r, x_j) = 0$ for all $0 \leqq i \leqq k-1$, $1 \leqq j \leqq n$, $2k < r \leqq n$.

We will presently verify the following statements about an irreducible Lie algebra L of linear transformations of V:

(a) If $1 \leqq k < n$ and $E_{1i} \in L$ for all $1 \leqq i \leqq k$ then there is a basis $\{y_1, \cdots, y_n\}$ of V such that $F_{1i} \in L$ for all $1 \leqq i \leqq k+1$.

(b) If $1 \leqq k < n$, $E_{1i} \in L$ for all $1 \leqq i \leqq n$, and $E_{i1} \in L$ for all $1 \leqq i \leqq k$ then there is a basis $\{y_1, \cdots, y_n\}$ of V such that $F_{1i} \in L$ for all $1 \leqq i \leqq n$ and $F_{i1} \in L$ for all $1 \leqq i \leqq k+1$.

(c) If $2 \leqq k \leqq m \leqq n$, $E_{1i} \in L$ for all $2 \leqq i \leqq m$, $E_{i1} \in L$ for all $2 \leqq i < k$, and L satisfies Condition $\sim$C then there is a basis $\{y_1, \cdots, y_n\}$ of V such that $F_{1i} \in L$ for all $2 \leqq i \leqq m$ and $F_{i1} \in L$ for all $2 \leqq i \leqq k$.

(d) If $3 \leqq k < n$ and E_{1i}, $E_{i1} \in L$ for all $2 \leqq i \leqq k$ then there is a basis $\{y_1, \cdots, y_n\}$ of V such that $F_{1i} \in L$ for all $2 \leqq i \leqq k+1$ and $F_{i1} \in L$ for all $2 \leqq i \leqq k$.

(e) If $\mathrm{sp}(x_1, \cdots, x_{2k}) \subseteq L$ where $\dim V > 2k \geqq 2$ and L satisfies Conditions $\sim$B and $\sim$C then $\dim V \geqq 2k+2$ and there is a basis $\{y_1, \cdots, y_n\}$ of V such that $\mathrm{sp}(y_1, \cdots, y_{2k+2}) \subseteq L$.

(f) If $\mathrm{sp}(V) \subseteq L$ and L satisfies Condition $\sim$B then $L = \mathrm{sp}(V)$ or $L = \mathrm{csp}(V)$.

Lemma 2 follows immediately from statements (a)–(f). For if Condition A holds we may choose a basis $\{x_1, \cdots, x_n\}$ for V such that $E_{11} \in L$. Then by (a), (b), and induction on k we see that $L = \mathrm{gl}(V)$, proving (i). If Conditions $\sim$A, B, and $\sim$C hold we may choose a basis $\{x_1, \cdots, x_n\}$ of V such that $E_{12} \in L$ and if $n \geqq 3$ we may also arrange that $E_{13} \in L$. Then using (c), (d), and induction on k we see that $L = \mathrm{sl}(V)$, proving (ii). Finally if Conditions $\sim$A, $\sim$B, and $\sim$C hold we may choose a basis $\{x_1, \cdots, x_n\}$ of V such that $E_{12} \in L$. By (c) we may assume that $\mathrm{sp}(x_1, x_2) \subseteq L$. Then by (e) and induction on k we have $\dim V = 2r \geqq 4$ and $\mathrm{sp}(V) \subseteq L$. Then (f) proves (iii).

We now verify (a)–(f). Throughout we will let $A = \sum E_{ij} a_{ij}$ where the $a_{ij} \in \Phi$.

(a): As $\langle x_1, \cdots, x_k \rangle$ is not an invariant subspace there exists $A \in L$ such that $a_{ij} \neq 0$ for some $1 \leqq i \leqq k < j \leqq n$. We may assume that $i = 1$ (replacing A by $A(\mathrm{ad}\, E_{1i})$ if $i \neq 1$) and that $a_{rj} = 0$ whenever $r > 1$ or $j = 1$ (replacing A by $(A(\mathrm{ad}\, E_{11}) - A)(\mathrm{ad}\, E_{11})/2$). Letting $\{y_1, \cdots, y_n\}$ be any basis for V satisfying $y_i = x_i$ for $1 \leqq i \leqq k$, $y_{k+1} = x_1 A$, and $y_j \in \langle x_{k+1}, \cdots, x_n \rangle$ for $j > k+1$ gives the result.

The proof of (b) is similar to that of (a).

(c): First assume that $\mathrm{ad}\, E_{1k} = 0$. Then for every $A \in L$ we have $0 = A(\mathrm{ad}\, E_{1k}) = \sum_i (E_{ik} a_{i1} - E_{1i} a_{ki})$. Hence $a_{ki} = 0$ for all $i \neq k$ so $\langle x_k \rangle$ is an invariant subspace, contradicting the irreducibility of L. Hence $\mathrm{ad}\, E_{1k} \neq 0$ so by Condition $\sim$C we have $(\mathrm{ad}\, E_{1k})^2 \neq 0$. Since $A(\mathrm{ad}\, E_{1k})^2 = E_{1k}(-2a_{k1})$ we have $E_{1k} \in L(\mathrm{ad}\, E_{1k})^2$. Since $E_{1k}^2 = 0$ we have $(\mathrm{ad}\, E_{1k})^3 = 0$. Thus Lemma V.8.2 of [8] shows that there exists $A \in L$ such that $A(\mathrm{ad}\, E_{1k})^2 = 2E_{1k}$ and $A(\mathrm{ad}\, E_{1k})(\mathrm{ad}\, A) = -2A$. Now

$$A(\mathrm{ad}\, E_{1k})(\mathrm{ad}\, A) = \sum_{i,j} E_{ij} \left(2a_{kj} a_{i1} - \delta_{i1} \sum_r a_{kr} a_{rj} - \delta_{jk} \sum_r a_{ir} a_{r1} \right).$$

Thus we have $a_{k1}=-1$ and

$$a_{ij} = -a_{i1}a_{kj} + \left(\delta_{i1}\sum_r a_{kr}a_{rj} + \delta_{jk}\sum_r a_{ir}a_{r1}\right)\Big/ 2. \tag{2}$$

Setting $i=j=1$ gives $0=\sum_r a_{kr}a_{r1}$. From this, using (2) to substitute for a_{rj} in $\sum_r a_{kr}a_{rj}$, we conclude that $\sum_r a_{kr}a_{rj}=0$ for all $1\leqq j\leqq n$ and similarly that $\sum_r a_{ir}a_{r1}=0$ for all $1\leqq i\leqq n$. Thus $a_{ij}=-a_{i1}a_{kj}$ and $\sum_r a_{ir}a_{rj}=0$ for $1\leqq i,\ j\leqq n$. Now set $y_1=x_kA$, $y_k=x_k$, and $y_j=x_j+a_{j1}x_k$ for $j\neq 1, k$. Then $F_{k1}=A\in L$, $F_{1k}=-E_{1k}\in L$, and $F_{1i}=-E_{1k}a_{i1}-E_{1i}\in L$ for $2\leqq i\leqq m$, $i\neq k$. Finally for $2\leqq i<k$ we have $F_{i1}=-E_{i1}(\operatorname{ad} F_{1k})(\operatorname{ad} F_{k1})-(F_{11}-F_{kk})a_{ki}\in L$ as required.

(d): As in (a) we may find a basis $\{y_1, \cdots, y_n\}$ of V such that F_{1i}, $F_{i1}\in L$ for $1\leqq i\leqq k$ and $A=\sum F_{ij}b_{ij}\in L$ where $b_{1j}=\delta_{j,k+1}$. Then

$$F_{1,k+1} = A(\operatorname{ad} F_{21})(\operatorname{ad} F_{12})(\operatorname{ad} F_{31})(\operatorname{ad} F_{13}) + (F_{11}-F_{33})b_{22}\in L$$

as required.

(e): It is well known (for example [**8**, p. 67] where we take $x_{2i+1}=v_{i+1}$ and $x_{2i+2}=v_{r+i+1}$ for $0\leqq i\leqq r-1$) that $\operatorname{sp}(x_1, \cdots, x_{2r})$ is spanned by the following elements and their transposes: $E_{2i+1,2i+1}-E_{2i+2,2i+2}$, $E_{2i+1,2i+2}$, $E_{2i+1,2j+1}-E_{2j+2,2i+2}$, and $E_{2i+1,2j+2}+E_{2j+1,2i+2}$ for $0\leqq i\neq j\leqq r-1$. Thus it is easily checked that $\operatorname{sp}(x_1, \cdots, x_{2k+2})$ is generated by $\operatorname{sp}(x_1, \cdots, x_{2k})$, $E_{1,2k+1}-E_{2k+2,2}$, and $E_{2k+1,2k+2}$.

We first show that for some basis $\{y_1, \cdots, y_n\}$ of V we have $\operatorname{sp}(y_1, \cdots, y_{2k})\subseteq L$ and $A=F_{1,2k+1}+\sum_{i>2k}F_{i2}b_{i2}\in L$ where the $b_{i2}\in\Phi$. As $\langle x_1, \cdots, x_{2k}\rangle$ is not an invariant subspace there exists $A\in L$ such that $a_{ij}\neq 0$ for some $1\leqq i\leqq 2k<j\leqq n$. We may assume that $i=1$ (replacing A by $A(\operatorname{ad} E_{12})$ if $i=2$, by $A(\operatorname{ad}(E_{1,2r+1}-E_{2r+2,2}))$ if $i=2r+1$, and by $A(\operatorname{ad}(E_{1,2r+2}+E_{2r+1,2}))$ if $i=2r+2$), that $a_{rs}=0$ unless $r=1$, $s>2$ or $r>2$, $s=2$ (replacing A by $A(\operatorname{ad} E_{21})(\operatorname{ad} E_{12})-(E_{11}-E_{22})(a_{11}-a_{22})-E_{12}(2a_{12})$), and that $a_{1s}=a_{s2}=0$ for $s\leqq 2k$ (replacing A by

$$A-\sum_{i=1}^{k-1} A((\operatorname{ad} E_{2i+2,2i+1})(\operatorname{ad} E_{2i+1,2i+2})+(\operatorname{ad} E_{2i+1,2i+2})(\operatorname{ad} E_{2i+2,2i+1})).$$

Then letting $\{y_1, \cdots, y_n\}$ be any basis for V satisfying $x_i=y_i$ for $1\leqq i\leqq 2k$, $y_{2k+1}=x_1A$, and $y_j\in\langle x_{2k+1}, \cdots, x_n\rangle$ for $j>2k+1$ gives the result.

Now assume that $\operatorname{sp}(x_1, \cdots, x_{2k})\subseteq L$ and that

$$A = E_{1,2k+1} + \sum_{i>2k} E_{i2}a_{i2}\in L.$$

We will show that for some basis $\{y_1, \cdots, y_n\}$ we have $\mathrm{sp}(y_1, \cdots, y_{2k}) \subseteq L$ and $F_{1,2k+1} - F_{2k+2,2} \in L$. If $a = a_{2k+1,2} \neq 0$ setting $y_i = x_i$ for $1 \leqq i \leqq 2k+1$ and $y_j = x_j - a^{-1}a_{j2}x_{2k+1}$ for $j > 2k+1$ we see that $\mathrm{sp}(y_1, \cdots, y_{2k}) \subseteq L$ and $A = F_{1,2k+1} + F_{2k+1,2}a$. Then $F_{21}(\mathrm{ad}\ A)^3 = (F_{1,2k+1} - F_{2k+1,2}a)3a \in L$ so $F_{1,2k+1} \in L$. This contradicts Condition $\sim$B so we must have $a = 0$. Also by Condition $\sim$B some $a_{i2} \neq 0$. Hence we can choose $y_{2k+2} \in \langle x_{2k+2}, \cdots, x_n \rangle$ so that $y_{2k+2}A = -x_2$. Then setting $y_i = x_i$ for $1 \leqq i \leqq 2k+1$ and choosing $y_j \in \langle x_{2k+2}, \cdots, x_n \rangle \cap \ker A$ for $j > 2k+2$ so that $\{y_1, \cdots, y_n\}$ is a basis for V we have the result.

Now assume $\mathrm{sp}(x_1, \cdots, x_{2k}) \subseteq L$ and $E_{1,2k+1} - E_{2k+2,2} \in L$. We will show that for some basis $\{y_1, \cdots, y_n\}$ of V we have $\mathrm{sp}(y_1, \cdots, y_{2k}) \subseteq L$ and $F_{1,2k+1} - F_{2k+2,2}$, $F_{2k+1,2k+2} \in L$, thus proving (e). We have $E_{2k+2,2k+1} = E_{21}(\mathrm{ad}(E_{1,2k+1} - E_{2k+2,2}))^2/2 \in L$. Then as in (c) we have $B = \sum E_{ij}b_{ij} \in L$ where the $b_{ij} \in \Phi$ and satisfy $b_{2k+1,2k+2} = -1$, $b_{ij} = -b_{i,2k+2}b_{2k+1,j}$, and $\sum_r b_{ir}b_{rj} = 0$ for $1 \leqq i, j \leqq n$. Then if $y_{2k+2} = -x_{2k+1}B$, $y_{2k+1} = x_{2k+1}$, and $y_j = x_j + b_{j,2k+2}x_{2k+1}$ for $j \neq 2k+1$, $2k+2$ we have $F_{2k+1,2k+2} = -B \in L$, $F_{2k+2,2k+1} = E_{2k+2,2k+1} \in L$, and $F_{1,2k+1} - F_{2k+2,2} = E_{1,2k+1} - E_{2k+2,2} - E_{2k+2,2k+1}(b_{2,2k+2} - b_{2k+1,1}) \in L$. Also, for $1 \leqq i, j \leqq 2k$, we have

$$F_{ij} = E_{ij} - E_{ij}(\mathrm{ad}\ F_{2k+1,2k+2})(\mathrm{ad}\ F_{2k+2,2k+1}) - F_{2k+2,2k+1}b_{ji}.$$

Hence $\mathrm{sp}(y_1, \cdots, y_{2k}) \subseteq L$ and $F_{1,2k+1} - F_{2k+2,2}$, $F_{2k+1,2k+2} \in L$ as required.

(f): If $\dim V = 2k$ and $\mathrm{sp}(V) \subseteq L$ then for $0 \leqq i \neq j \leqq k-1$ if $A \in L$ then

$$\begin{aligned} &E_{2i+1,2j+1}a_{2i+1,2j+1} + E_{2j+2,2i+2}a_{2j+2,2i+2} \\ &\quad = A(\mathrm{ad}\ E_{2i+2,2i+1})(\mathrm{ad}\ E_{2i+1,2i+2})(\mathrm{ad}\ E_{2j+1,2j+2})(\mathrm{ad}\ E_{2j+2,2j+1}) \in L. \end{aligned}$$

Then by Condition $\sim$B we must have $a_{2i+1,2j+1} + a_{2j+2,2i+2} = 0$. Similarly we see that $a_{2i+1,2j+2} = a_{2j+1,2i+2}$ and $a_{2i+2,2j+1} = a_{2j+2,2i+1}$. Hence $A = D + S$ where $S \in \mathrm{sp}(V)$ and $D = \mathrm{diag}\{d_1, \cdots, d_{2k}\} \in L$. Now

$$D(\mathrm{ad}(E_{1,2i+1} - E_{2i+2,2})) = E_{1,2i+1}(d_1 - d_{2i+1}) - E_{2i+2,2}(d_{2i+2} - d_2) \in L.$$

Thus, again by Condition $\sim$B, we have $d_1 - d_{2i+1} = d_{2i+2} - d_2$ for all $0 \leqq i \leqq k-1$. Thus $2D = I(d_1 + d_2) + E$ where $E \in \mathrm{sp}(V)$. Thus $L = \mathrm{sp}(V)$ or $L = \mathrm{csp}(V)$.

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