

ON DETERMINATION OF THE OPTIMAL FACTOR OF A NONNEGATIVE MATRIX-VALUED FUNCTION

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ABSTRACT. Let $F = [f_{ij}]$, $1 < i, j \leq q$, be a measurable, nonnegative definite $q \times q$ matrix-valued function defined on the unit circle C . It is known that when F and $\log \det F$ are in $L_1(C)$, F admits a factorization of the form $F = \Phi \Phi^*$, where Φ is an optimal, full rank function in $L_2^{0+}(C)$. Under the additional assumption that $\{(\prod_{i=1}^q f_{ii}) / \det F\}$ is in $L_1(C)$, an iterative procedure which yields an infinite series for Φ in terms of F is given. The optimal function Φ plays a significant role in the multivariate prediction theory of stochastic processes. The present work generalizes the results of several authors concerning the determination of the optimal factor Φ .

1. Introduction. Let $F = [f_{ij}]$, $1 \leq i, j \leq q < \infty$, be a measurable, nonnegative definite $q \times q$ matrix-valued function defined on the unit circle C . An important problem in multivariate prediction theory is, given that $F \in L_1$ and $\log \det F \in L_1$, to find a measurable $q \times q$ matrix-valued function Φ defined on C such that $F = \Phi \Phi^*$ a.e. on C , $\Phi \in L_2^{0+}$, whose 0th Fourier coefficient C_0 has the property that

$$C_0 > 0 \quad \text{and} \quad \det C_0 = \exp \frac{1}{2\pi} \int_0^{2\pi} \log \det F \, d\theta > 0.$$

An interesting iterative procedure which yields an infinite series for Φ in terms of F was given by Wiener and Masani in [5] under the hypothesis that there exist constants c_1, c_2 ; $0 \leq c_1 \leq c_2 < \infty$, such that:

$$(1.1) \quad c_1 I \leq F \leq c_2 I.$$

In [2] Masani was able to improve the results he and Wiener gave in [5] by assuming in lieu of condition (1.1) that:

$$(1.2) \quad \begin{array}{ll} \text{(i)} & F \in L_1, \\ \text{(ii)} & F^{-1} \text{ exists a.e. on } C \text{ and } F^{-1} \in L_1, \\ \text{(iii)} & \text{if } \lambda \text{ and } \mu \text{ denote the smallest and} \\ & \text{largest eigenvalues of } F, \text{ then } \mu/\lambda \in L_1. \end{array}$$

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In his work [2] Masani repeatedly made use of the fact that $F^{-1} \in L_1$. However for many situations this condition is not necessarily satisfied as was pointed out by the author in [4].

The purpose here is to give an algorithm for finding Φ , similar to the one obtained in [5], [2] and [4], under a weaker and a more readily verifiable assumption on F . The main result of this paper is:

1.3 THEOREM. Let $F = [f_{ij}]$, $1 \leq i, j \leq q$, be a measurable, nonnegative $q \times q$ matrix-valued function on C such that

$$(1.4) \quad \begin{aligned} & \text{(i)} \quad F \in L_1 \quad \text{and} \quad \log \det F \in L_1, \\ & \text{(ii)} \quad \left\{ \left(\prod_{i=1}^q f_{ii} \right) / \det F \right\} \in L_1. \end{aligned}$$

Then

- (a) $f_{ii} = \phi_i \bar{\phi}_i$, $\phi_i \in L_2^{0+}$ (see §2), $|\phi_i| \neq 0$ a.e. on C , $1 \leq i \leq q$.
- (b) Let $\hat{F} = [\hat{f}_{ij}]$, $\hat{f}_{ij} = f_{ij} / (\phi_i \bar{\phi}_j)$, $1 \leq i, j \leq q$, and λ, μ denote the smallest and the largest eigenvalues of \hat{F} . If $f = \frac{1}{2} \{ \mu + \lambda \}$, then $f = \phi \bar{\phi}$, $\phi \in L_2^{0+}$, $|\phi| \neq 0$ a.e. on C .
- (c) Let M be defined by $\hat{F} = f(I + M)$. Then $I + M = \alpha \alpha^*$, $\alpha \in L_2^{0+}$, α optimal of full rank in L_2^{0+} (see Definition 2.1).
- (d) F is factorable, $F = \Phi \Phi^*$, $\Phi \in L_2^{0+}$, Φ optimal of full rank in L_2^{0+} . If Σ is defined by

$$\Sigma = \begin{bmatrix} \phi_1 & & & & & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \phi_i & & \\ & & & & \cdot & \\ 0 & & & & & \cdot \\ & & & & & & \phi_q \end{bmatrix},$$

then the optimal factor Φ is given by $\Phi = \phi \Sigma \alpha$.

The proof of this theorem essentially consists of an initial factorization of the diagonal entries of F , followed by the application of the known results to the remaining factor. After determining the factors ϕ , Σ and α in §2 and §3, a proof of Theorem 1.3 will be given near the end of §3.

1.5 REMARK. It is easy to see that (1.1) implies (1.4). We will show that the condition $(\mu/\lambda)^{q-1} \in L_1$ implies (1.4) (ii) (for $q=2$ this condition reduces to (1.2) (iii)), and that (1.2) (ii) is not necessary for the determination of Φ . We will also give a simple example where (1.2) (ii), (iii) fail to hold, yet (1.4) is satisfied.

2. Preliminary results. As in [2] bold face letters A, B , etc. will

denote $q \times q$ matrices and bold face letters F, Φ , etc. will denote functions whose values are such matrices. tr , \det , $*$, will be reserved for the trace, determinant and adjoint of matrices. $|A|_B$, will denote the Banach norms of A .

We shall be concerned with the class L_p ($1 \leq p \leq \infty$) of $q \times q$ matrix-valued functions F on C whose entries are in L_p in the usual sense. $L_2^+, L_2^{0+}, L_2^-, L_2^{0-}$ will denote the subspace of functions in L_2 whose n th Fourier coefficients vanish for $n \leq 0, n < 0, n \geq 0, n > 0$, respectively. If $F \in L_2$ and has Fourier coefficients $A_k, -\infty < k < \infty$, then F_+, F_{0+}, F_-, F_{0-} , will denote the functions in $L_2^+, L_2^{0+}, L_2^-, L_2^{0-}$, whose n th Fourier coefficients are A_n , for $n > 0, n \geq 0, n < 0, n \leq 0$, respectively (and zero for the remaining n). In L_2 we introduce the Gramian, inner product and norm

$$(\Phi, \Psi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(e^{i\theta}) \Psi^*(e^{i\theta}) d\theta, \quad ((\Phi, \Psi)) = \text{tr}(\Phi, \Psi),$$

$$\|\Phi\| = (\text{tr}(\Phi, \Phi))^{1/2}.$$

2.1 DEFINITION. (a) $\Phi \in L_2$ is said to be of full-rank if $|\det \Phi(e^{i\theta})| > 0$ a.e. on C . (b) Φ is called an optimal function in L_2^{0+} if $\Phi \in L_2^{0+}, \Phi(0)^2 \geq 0$ and $\Psi \in L_2^{0+}, \Psi \Psi^* = \Phi \Phi^*$ implies $\{\Psi(0) \Psi^*(0)\}^{1/2} \leq \Phi(0)$.

§4 of [2] contains several interesting results pertaining to the determination of the generating function of a multivariate stochastic process. Below we summarize those results which are mostly relevant to our work as will be needed in §3.

2.2 THEOREM. Let (i) M be a hermitian $q \times q$ matrix-valued function on C , and $M \in L_\infty$,

- (ii) $|M(e^{i\theta})|_B < 1$ a.e. on C ,
- (iii) $(I + M)^{-1} \in L_1$,
- (iv) for each $\Psi \in L_2, \mathcal{O}(\Psi) = (\Psi M)_+$.

Then (a) \mathcal{O} is a bounded linear operator on L_2 into L_2^+ .

(b) $\mathcal{O}(I) = M_+, \mathcal{O}^2(I) = (M_+ M)_+$, and so on.

(c) The series $I - M_+ + (M_+ M)_+ - \{(M_+ M)_+ M\}_+ + \dots$ is mean convergent.

(d) If χ is the optimal factor of $(I + M)^3$ with C_0 as its 0th Fourier coefficient, Ψ the sum of the series in (c), then

$$(2.3) \quad \Psi = C_0 \chi^{-1} \in L_2^{0+}, \quad C_0^2 = \Psi(I + M) \Psi^*.$$

² For a function $\Phi \in L_2^{0+}, \Phi(0)$ denotes the value of the analytic extension of Φ into the interior of C . Obviously $\Phi(0)$ is the 0th Fourier coefficient of F .

³ We note that under our conditions on $M, I + M$ is factorable.

In (3.4) the first and the third matrices on the right are in L_2^{0+} and L_2^{0-} , and the one in the middle, \hat{F} , is well defined and is in L_∞ . Moreover \hat{F} is nonnegative and $\log \det \hat{F} \in L_1$. In brief

$$(3.5) \quad \hat{F} \in L_\infty, \quad \log \det \hat{F} \in L_1.$$

By (3.5) \hat{F} is factorable:

$$(3.6) \quad \hat{F} = \Omega \Omega^*, \quad \Omega \in L_\infty^{0+}, \quad \Omega(0) > 0, \quad \Omega \text{ optimal in } L_2^{0+}.$$

From (3.4) and (3.6) we obtain

$$(3.7) \quad F = (\Sigma \Omega)(\Sigma \Omega)^*.$$

Now let $\Phi = \Sigma \Omega$. Since $\Sigma \in L_2^{0+}$ and $\Omega \in L_\infty^{0+}$, we conclude that

$$(3.8) \quad \Phi = \Sigma \Omega \in L_2^{0+}.$$

Moreover since Σ and Ω are full-rank, optimal factors by Lemma 2.3 [4] it follows that Φ is the full-rank, optimal factor of F .

In view of these results, for the determination of Φ , it suffices to determine the Fourier coefficients of the optimal factor Ω of \hat{F} .

3.9 THEOREM. Let $F = [f_{ij}]$, $1 \leq i, j \leq q$, be a measurable nonnegative $q \times q$ matrix-valued function satisfying condition (5.1), and let \hat{F} be defined as in (3.4). Then

- (a) $\hat{F} \in L_\infty \subseteq L_1$.
- (b) $\hat{F}^{-1} \in L_1$.
- (c) If λ, μ denote the smallest and largest eigenvalues of \hat{F} , then $\mu/\lambda \in L_1$.

PROOF. (a) Since $|f_{ij}/\phi_i \bar{\phi}_j| \leq 1$, $\hat{f}_{ij} = f_{ij}/\phi_i \bar{\phi}_j$, the (i, j) th entry of \hat{F} is in L_∞ and hence $\hat{F} \in L_\infty$.

(b) Since the entries of \hat{F} are bounded by 1 it follows that the entries of \hat{F}^{-1} are bounded by $\{(q-1)!\}^2/\det \hat{F}$.⁴ But

$$1/\det \hat{F} = \left(\prod_{i=1}^q f_{ii} \right) / \det F \in L_1.$$

Therefore $\hat{F}^{-1} \in L_1$.

(c) follows from (a) and (b) (see [2, p. 158]).

Now, as in [2], we let

$$(3.10) \quad f = \frac{1}{2} \{ \mu + \lambda \}, \quad M = \frac{1}{f} \hat{F} - I.$$

⁴ One can show that this bound can be replaced by $1/\det \hat{F}$.

Then as shown in [2], we have

$$(3.11) \quad \begin{aligned} & \text{(i)} \quad \hat{F} = f(I + M), \\ & \text{(ii)} \quad |M|_B < 1 \text{ a.e. on } C, \\ & \text{(iii)} \quad I + M, (I + M)^{-1} \in L_1, \\ & \text{(iv)} \quad f, 1/f \in L_1. \end{aligned}$$

By (3.11) (iv), since f is scalar valued, its optimal factor ϕ can be found and we have

$$(3.12) \quad f = \phi\bar{\phi}, \quad \phi \in L_2^{0+}, \quad |\phi| \neq 0 \text{ a.e. on } C.$$

By (3.11) (ii)–(iii) and Theorem 2.2 the optimal factor α of $I + M$ can be found by an iterative procedure (cf. (2.3)–(2.6)). We write

$$(3.13) \quad I + M = \alpha\alpha^*, \quad \alpha \in L_2^{0+}, \quad \alpha(0) > 0, \quad \alpha \text{ optimal.}$$

Summing up we have proved our main result Theorem 1.3.

The following lemma and corollary shed some light on the relation between the hypothesis (1.2) (iii) made in [2] and our condition (1.4).

3.14 LEMMA. *Let $F = [f_{ij}]$, $1 \leq i, j \leq q$, be a measurable, nonnegative $q \times q$ matrix-valued function on C . Let λ, μ be the smallest and largest eigenvalues of F . If $(\mu/\lambda)^{q-1} \in L_1$, then $(\prod_{i=1}^q f_{ii})/\det F \in L_1$.*

3.15 COROLLARY. *Let $q = 2$. Then hypothesis (1.2) (iii) alone implies our condition (1.4) (ii).*

Next we give an example for which the hypothesis (1.4) (ii) is satisfied, but condition (1.2) (ii)–(iii) is not satisfied.

3.16 EXAMPLE. Let $F = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}$, where

- (i) $0 \leq f, g \in L_1; \log f, \log g \in L_1$,
- (ii) $1/f, 1/g \notin L_1$,
- (iii) $(g/f) \notin L_1$.⁵

Obviously (1.2) (ii)–(iii) are not satisfied, but (1.4) (ii) is satisfied, because

$$\left(\prod_{i=1}^2 f_{ii} \right) / \det F = fg/fg = 1 \in L_1.$$

We note that our algorithm remains valid with $\hat{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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⁵ Take for example $f = \exp\{-1/|\theta - 1|^\alpha\}$, $g = \exp\{-1/|\theta + 1|^\alpha\}$, $0 < \alpha < 1$.

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