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SHORTER NOTES

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ROTATIONAL APPROXIMATION

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The purpose of this note is to give a short, "soft" proof of a strengthened version of the rotational completeness theorem [1], and to state a generalization of it for analytic functions on an annulus.

The strengthened version of the rotational completeness theorem reads as follows:

THEOREM. *Suppose that* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *and* $g(z)=\sum_{n=0}^{\infty} b_n z^n$ *are analytic in* $|z|<1$ *and that the set* $\{e^{i\theta_m} \mid 0\leqq\theta_m<2\pi,\ m\in P\}$ *is infinite. Then there exists a sequence* $f_n(z)=\sum_{p=0}^{n} a_{n_p} f(e^{i\theta_p}z)$, $p=0, 1, \cdots, n$ *and* $n=1, 2, \cdots$, *such that* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly in compact subsets to g on* $|z|<1$ *iff* $b_n=0$ *whenever* $a_n=0$, $n=0, 1, \cdots$.

PROOF. The necessity is clear. For the sufficiency, let F be a linear functional on the convex linear topological space A of all analytic functions on $|z|<1$ with the compact-open topology which vanishes on the linear subspace generated by $\{f(e^{i\theta_m}z) \mid m\in P\}$.

It is known, for example in [1], that F is of the form that $F(f)=\sum_{n=0}^{\infty} c_n a_n$, where $\limsup_{n\to\infty} |c_n|^{1/n}<1$.

Now the domain of analyticity of the function $h(z)=\sum_{n=0}^{\infty} c_n a_n z^n$ contains the closed disk and the function $h(z)$ vanishes on an infinite set $\{e^{i\theta_m} \mid m\in P\}$. Hence $a_n c_n=0$, $n=0, 1, \cdots$. The condition that $b_n=0$ whenever $a_n=0$, $n=0, 1, \cdots$, implies that $c_n b_n=0$, $n=0, 1, \cdots$, and hence $F(g)=0$. By a well-known fact for a convex topological space, the sufficiency follows.

It is quite easy to find out the explicit form of the continuous linear functionals on the space of all analytic functions on an annulus. With the same idea, one obtains the following:

THEOREM. *Suppose that* $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n$ *and* $g(z)=\sum_{n=-\infty}^{\infty} b_n z^n$ *are analytic in the annulus* $1/R<|z|<R$, *and that the set*

$$\{e^{i\theta_m} \mid 0\leqq\theta_m<2\pi,\ m\in P\}$$

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is infinite. Then the closure of the linear subspace generated by $\{f(e^{i\theta_m}z) \mid m \in P\}$ *contains* g *iff* $b_n = 0$ *whenever* $a_n = 0$, $n = 0, \pm 1, \pm 2, \cdots$.

References

1. Pasquale Porcelli, *Linear spaces of analytic functions*, Rand McNally, Chicago, Ill., 1966.

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